Duality in left-right symmetric seesaw

Evgeny Akhmedov

KTH, Stockholm & Kurchatov Institute, Moscow

In collaboration with Michele Frigerio

Why are neutrinos so light ?

♢ A simple and elegant explanation – The seesaw mechanism

(Minkowski, 1977; Gell-Mann, Ramond & Slansky, 1979; Yanagida, 1979; Glashow, 1979; Mohapatra & Senjanović, 1980)

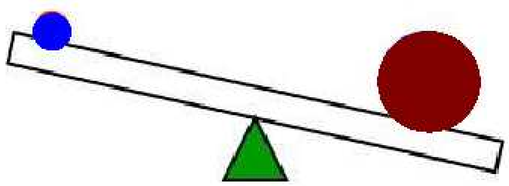
In addition:

Seesaw has a built-in mechanism for generating the baryon asymmetry of the Universe – Baryogenesis via Leptogenesis

(Fukugita & Yanagida, 1986; Luty, 1992; Covi et al., 1996; Buchmüller & Plümacher, 1996; ...)

The simplest version: Add 3 RH ν's N_{Ri} yo the minimal SM

Heavy N_{Ri}'s make ν_{Li}'s light :



$$-\mathcal{L}_{Y+m} = Y_\nu \, \bar{l}_L \, N_R \, H + \frac{1}{2} M_R N_R N_R + h.c.,$$

Neutrino mass matrix in the $(\nu_L,\ (N_R)^c)$ basis:

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}$$

N_{Ri} are EW singlets $\Rightarrow$ $M_R \sim M_{GUT}(M_I) \gg m_D \sim v$.

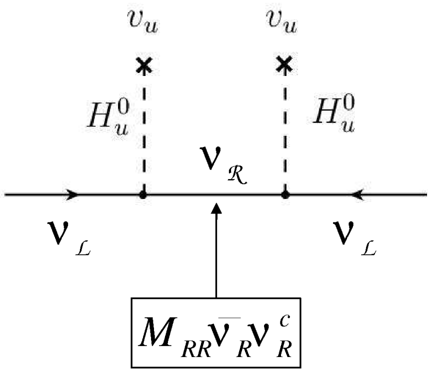
Block diagonalization: $M_N \simeq M_R$,

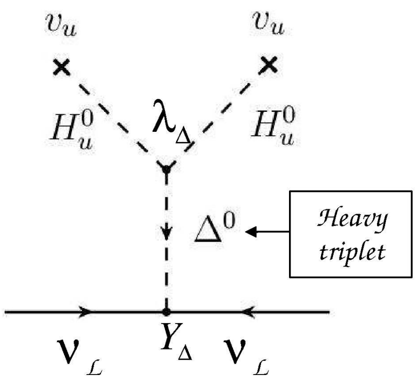
$\diamondsuit$ $\boxed{m_{\nu_L} \simeq -m_D \, M_R^{-1} \, m_D^T}$ $\Rightarrow$ $m_\nu \sim \frac{(174\ \mathrm{GeV})^2}{M_R}$

For $m_\nu \lesssim 0.05\ \mathrm{eV}$ $\Rightarrow$ $M_R \gtrsim 10^{15}\ \mathrm{GeV} \sim M_{GUT} \sim 10^{16}\ \mathrm{GeV}$!

Type II seesaw

Type I seesaw: Neutrino mass generated through exchage of a heavy RH ν's N_R; in type II – through exchage of a heavy $SU(2)_L$ - triplet scalars Δ_L:





Type II seesaw: Magg & Wetterich, 1980; Lazarides et al., 1981; Schechter & Valle, 1980; Mohapatra & Senjanović, 1981,. . .

In the SM, RH neutrinos N_R are singlets of the gauge group – “aliens”.
They are more natural in Left-Right symmetric extensions of the SM:
$SU(2)_L \times SU(2)_R \times U(1)_{\rm B-L}$, $SU(2)_L \times SU(2)_R \times SU(4)_{\rm PS}$, $SO(10)$, ...

LR symmetric models explain in a nice way maximal P- and C-violation in low-E weak interactions as a spontaneous symmetry breaking phenomenon
– likely to be present in the final theory

In LR - symmetric models:

In general, both type I and type II seesaw contributions to m_ν are present :

$$\mathcal{M}_\nu = \begin{pmatrix} m_L & m_D \\ m_D^T & M_R \end{pmatrix}$$

Block diagonalization of $\mathcal{M}_\nu$:

$$\diamondsuit \quad m_\nu \simeq m_L - m_D M_R^{-1} m_D^T \,, \qquad m_L = f_L v_L$$

m_D comes from Yukawa interactions with the bi-doublet Higgs Φ:

$$Y_1 \bar{l}_L \Phi l_R + Y_2 \bar{l}_L \tilde{\Phi} l_R \,, \quad \tilde{\Phi} \equiv \tau_2 \Phi^* \tau_2 \qquad \Rightarrow \qquad m_D = yv$$

m_L and M_R – from Yukawa couplings with triplet Higgses Δ_L and Δ_R

$$\Rightarrow \qquad m_\nu \simeq f_L v_L - v^2 y (f_R v_R)^{-1} y^T$$

In general: m_ν, f_L and f_R – complex symmetric, y – complex matrix

Discrete LR symmetry

Discrete LR symmetry in LR-symm. models – two possible implementations (P or C):

$$(1) \qquad l_{Li} \leftrightarrow l_{Ri}\,, \qquad \Delta_L \leftrightarrow \Delta_R^*\,, \qquad \Phi \leftrightarrow \Phi^\dagger$$

Yields

$$f_L = f_R^*\,, \qquad y = y^\dagger\,,$$

$$(2) \qquad l_{Li} \leftrightarrow l_{Li}^c \equiv (l_{Ri})^c\,, \quad \Delta_L \leftrightarrow \Delta_R\,, \qquad \Phi \leftrightarrow \Phi^T$$

Yields

$$f_L = f_R \equiv f\,, \qquad y = y^T\,.$$

Both implementations possible. Implem. (2) is more natural in $SO(10)$ GUTs (is an automatic gauge symmetry)

Adhere to the second implementation $\Rightarrow$

$$\boxed{m_\nu \;=\; f v_L \;-\; v^2\, y\, (v_R f)^{-1}\, y}$$

For realization (1) (parity): type I term contains $(f^*)^{-1}$ instead of f^{-1}.

N_R, Δ_R, Δ_L are all at very high scale ($\sim\; 10^{12} \;-\; 10^{16}\;\mathrm{GeV}$) $\Rightarrow$ no direct way of probing this sector of the theory.

$\Diamond$ Neutrinos may provide a low-energy window into new physics at very high energy scales !

Bottom-up approach:

- Take m_ν from experiment
- Take y from data + theoretical assumptions (quark-lepton symm., GUTs)
- Solve the seesaw relation for f

E.A. and M. Frigerio, PRL 96 (2006) 061802 and hep-ph/0609046

What would we like to know?

Try to learn as much as possible about the heavy neutrino sector

- What are the masses of heavy RH neutrinos?
- What are the mixing and CP violation in the heavy RH neutrino sector?
- Can all this give us a hint of the underlying theory?
- Is the solution that we found unique?
- Is type I or type II contribution in the light neutrino mass dominant or are they equally important?

Inverting the seesaw formula

Pure type I $(c_0 \equiv v^2/v_R)$:

$$m_\nu = -c_0 \left(y\, f^{-1}\, y^T\right) \qquad \Rightarrow \qquad f = -c_0 \left(y^T\, m_\nu^{-1}\, y\right)$$

Inverting the seesaw formula

Pure type I $(c_0 \equiv v^2/v_R)$:

$$m_\nu = -c_0 \left(y f^{-1} y^T\right) \qquad \Rightarrow \qquad f = -c_0 \left(y^T m_\nu^{-1} y\right)$$

Pure type II:

$$m_\nu = f v_L \qquad \Rightarrow \qquad f = m_\nu / v_L$$

Inverting the seesaw formula

Pure type I $(c_0 \equiv v^2/v_R)$:

$$m_\nu = -c_0 \left(y f^{-1} y^T\right) \quad \Rightarrow \quad f = -c_0 \left(y^T m_\nu^{-1} y\right)$$

Pure type II:

$$m_\nu = f v_L \quad \Rightarrow \quad f = m_\nu / v_L$$

General LR-symmetric type I + II seesaw:

$$m_\nu = f v_L - \frac{v^2}{v_R} \left(y f^{-1} y\right)$$

– a nonlinear matrix equation for f. Yet can be readily solved analytically !

Inverting the seesaw formula

Pure type I $(c_0 \equiv v^2/v_R)$:

$$m_\nu = -c_0 \left(y\, f^{-1}\, y^T\right) \qquad \Rightarrow \qquad f = -c_0 \left(y^T\, m_\nu^{-1}\, y\right)$$

Pure type II:

$$m_\nu = f\, v_L \qquad \Rightarrow \qquad f = m_\nu / v_L$$

General LR-symmetric type I + II seesaw:

$$m_\nu = f v_L - \frac{v^2}{v_R} \left(y\, f^{-1}\, y\right)$$

– a nonlinear matrix equation for f. Yet can be readily solved analytically !

Important point: A duality property of LR-symmetric seesaw

♢ If f is a solution, so is $\hat{f} \equiv (m_\nu/v_L - f)$

Inverting the seesaw formula

Pure type I $(c_0 \equiv v^2/v_R)$:

$$m_\nu = -c_0 \left(y\, f^{-1}\, y^T\right) \quad \Rightarrow \quad f = -c_0 \left(y^T\, m_\nu^{-1}\, y\right)$$

Pure type II:

$$m_\nu = f\, v_L \quad \Rightarrow \quad f = m_\nu / v_L$$

General LR-symmetric type I + II seesaw:

$$m_\nu = f v_L - \frac{v^2}{v_R} \left(y\, f^{-1}\, y\right)$$

– a nonlinear matrix equation for f. Yet can be readily solved analytically !

Important point: A duality property of LR-symmetric seesaw

♢ If f is a solution, so is $\hat{f} \equiv (m_\nu/v_L - f)$

$\Rightarrow$ There is always an even number of solutions! $(\#sol. = 2^n)$

LR symmetry important for duality!

It was important that $f_L = f_R$ and $y^T = y$.

◊ For the other realization of parity symmetry $(f_L = f_R^*$ and $y = y^\dagger)$ exactly the same duality holds !

Two-generation case

$$f^{-1} = \frac{1}{F}\begin{pmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{pmatrix}, \quad y = \begin{pmatrix} y_{e1} & y_{e2} \\ y_{\mu 1} & y_{\mu 2} \end{pmatrix},$$

$F \equiv \det f$, $y_{\mu 1} = y_{e2}$. Define: $x \equiv v_L v_R / v^2$ and $m \equiv m_\nu / v_L \quad \Rightarrow$

$$\begin{aligned}
xF(f_{11} - m_{ee}) &= f_{22}\, y_{e1}^2 - 2 f_{12}\, y_{e1}\, y_{e2} + f_{11}\, y_{e2}^2\,, \\
xF(f_{22} - m_{\mu\mu}) &= f_{22}\, y_{e2}^2 - 2 f_{12}\, y_{e2}\, y_{\mu 2} + f_{11}\, y_{\mu 2}^2\,, \\
xF(f_{12} - m_{e\mu}) &= f_{22}\, y_{e1} y_{e2} - f_{12}\,(y_{e1}\, y_{\mu 2} + y_{e2}^2) \\
&\quad + f_{11}\, y_{e2}\, y_{\mu 2}\,,
\end{aligned}$$

A system of coupled 3rd order equations

Admits a simple exact analytic solution !

Go to the basis where y is diagonal: $y = diag(y_1,\, y_2)$ (no loss of generality)

$$\begin{aligned} xF(f_{11} - m_{ee}) &= f_{22}\, y_1^2 \\ xF(f_{22} - m_{\mu\mu}) &= f_{11}\, y_2^2 \\ xF(f_{12} - m_{e\mu}) &= -\, f_{12}\, y_1\, y_2 \end{aligned}$$

Rescaling:

$$f_{ij} = \sqrt{\lambda}\, f'_{ij}\,, \quad m_{ij} = \sqrt{\lambda}\, m'_{ij}\,, \quad y_{ij} = \sqrt{\lambda}\, y'_{ij}\,,$$

λ - arbitrary; fix it by requiring $F' \equiv \det f' = 1$. The system of eqs. for f'_{ij} becomes linear

Express f'_{ij} back through unprimed $m_{ij},\, y_{1,2}$ and subst. into $F' = 1 \Rightarrow$ 4th order characteristic equation for λ

The solution:

$$f = \frac{x\lambda}{(x\lambda)^2 - y_1^2 y_2^2} \begin{pmatrix} x\lambda m_{ee} + y_1^2 m_{\mu\mu} & m_{e\mu}(x\lambda - y_1 y_2) \\ \ldots & x\lambda m_{\mu\mu} + y_2^2 m_{ee} \end{pmatrix}$$

λ has to be found from

$$\left[(x\lambda)^2 - y_1^2 y_2^2\right]^2 - x\left[\det m (x\lambda - y_1 y_2)^2 x\lambda \right.$$
$$\left. + (m_{ee} y_2 + m_{\mu\mu} y_1)^2 (x\lambda)^2\right] = 0$$

Has 4 complex solutions $\Rightarrow$ 4 solutions for f which form 2 dual pairs

Determinant of the seesaw relation $x\hat{f} = -y f^{-1} y \quad \Rightarrow \quad x^2 F\hat{F} = y_1^2 y_2^2$
$F' = 1 \quad \Rightarrow \quad F = \lambda$; therefore

$$x^2 \lambda\hat{\lambda} = y_1^2 y_2^2$$

Allows to express the four solutions for λ in a simple closed form

Which seesaw type dominates ?

If $|m_{\alpha\beta} m_{\gamma\delta}| \gg 4|y_i y_j / x|$, one of the solutions λ_i satisfies $|x\lambda_1| \gg |y_i y_j|$

$\Rightarrow$ $f_1 \simeq m$ (type II seesaw)

Which seesaw type dominates ?

If $|m_{\alpha\beta}m_{\gamma\delta}| \gg 4|y_i y_j/x|$, one of the solutions λ_i satisfies $|x\lambda_1| \gg |y_i y_j|$

$\Rightarrow$ $f_1 \simeq m$ (type II seesaw)

Then for the dual solution λ_2: $|x\lambda_2| \ll |y_i y_j|$

$\Rightarrow$ $f_2 = \hat{f}_1$ corresponds to type I seesaw

The remaining 2 solutions are of mixed type.

Which seesaw type dominates ?

If $|m_{\alpha\beta}m_{\gamma\delta}| \gg 4|y_i y_j/x|$, one of the solutions λ_i satisfies $|x\lambda_1| \gg |y_i y_j|$

$\Rightarrow$ $f_1 \simeq m$ (type II seesaw)

Then for the dual solution λ_2: $|x\lambda_2| \ll |y_i y_j|$

$\Rightarrow$ $f_2 = \hat{f}_1$ corresponds to type I seesaw

The remaining 2 solutions are of mixed type.

If $|m_{\alpha\beta}m_{\gamma\delta}| \lesssim 4|y_i y_j/x|$, all 4 solutions are of mixed type.

Similar situation in 3-generation case

Which seesaw type dominates ?

If $|m_{\alpha\beta} m_{\gamma\delta}| \gg 4|y_i y_j / x|$, one of the solutions λ_i satisfies $|x\lambda_1| \gg |y_i y_j|$

$\Rightarrow$ $f_1 \simeq m$ (type II seesaw)

Then for the dual solution λ_2: $|x\lambda_2| \ll |y_i y_j|$

$\Rightarrow$ $f_2 = \hat{f}_1$ corresponds to type I seesaw

The remaining 2 solutions are of mixed type.

If $|m_{\alpha\beta} m_{\gamma\delta}| \lesssim 4|y_i y_j / x|$, all 4 solutions are of mixed type.

Similar situation in 3-generation case

In general, if there is a solution with seesaw type I dominance, there is always also a solution with type II sessaw dominance and vice versa.

Which seesaw type dominates ?

If $|m_{\alpha\beta}m_{\gamma\delta}| \gg 4|y_i y_j/x|$, one of the solutions λ_i satisfies $|x\lambda_1| \gg |y_i y_j|$

$\Rightarrow$ $f_1 \simeq m$ (type II seesaw)

Then for the dual solution λ_2: $|x\lambda_2| \ll |y_i y_j|$

$\Rightarrow$ $f_2 = \hat{f}_1$ corresponds to type I seesaw

The remaining 2 solutions are of mixed type.

If $|m_{\alpha\beta}m_{\gamma\delta}| \lesssim 4|y_i y_j/x|$, all 4 solutions are of mixed type.

Similar situation in 3-generation case

In general, if there is a solution with seesaw type I dominance, there is always also a solution with type II sessaw dominance and vice versa.

Using low-energy data (and v_R) only, it is impossible to say which seesaw type dominataes (if any) – additional criteria necessary (leptogenesis?)

Three lepton generations

A system of six coupled 4th order equations – can be readily solved analytically by making use of the same trick as in 2-g case plus duality.

Characteristic equation for the rescaling factor λ is of 8th order $\Rightarrow$
8 solutions λ_i $\Rightarrow$ 8 solutions f_i $(i = 1, \ldots, 8)$.

Form 4 pairs of mutually dual solutions

For given y (i.e. m_D), all 8 solutions result in exactly the same mass matrix of light neutrinos m_ν !

Analysis of solutions recently done: Hosteins et al. (hep-ph/0606078);
EA & Frigerio (hep-ph/0609046)

Study of some stability and leptogenesis issues under way (Stockholm group)

A realistic numerical example

Take: normal hierarchy, tri-by-maxial mixing, no $\not{CP}$,

$$m_1 = 0.005\,\text{eV}, \quad \Delta m^2_{21}/\Delta m^2_{31} = 0.031, \quad v_L \simeq 0.05\,\text{eV}, \quad v_R \simeq 6\times 10^{14}\,\text{GeV}$$

Also: $V_{\text{CKM}} \approx \mathbb{1}, \quad y_1 = 0.01, \quad y_2 = 0.1, \quad y_3 = 1$

$$\Downarrow$$

$$m \equiv \frac{m_\nu}{v_L} = \begin{pmatrix} 0 & 0.1 & -0.1 \\ 0.1 & 0.55 & 0.45 \\ -0.1 & 0.45 & 0.55 \end{pmatrix}$$

Now solve the seesaw equation for f

Solutions:

$$f_1 \approx \begin{pmatrix} -0.001 & 0.10 & -0.14 \\ \dots & 0.56 & 0.49 \\ \dots & \dots & 0.88 \end{pmatrix} \qquad \hat{f}_1 \approx \begin{pmatrix} 0.001 & -0.005 & 0.04 \\ \dots & -0.006 & -0.04 \\ \dots & \dots & -0.33 \end{pmatrix}$$

$$f_2 \approx \begin{pmatrix} -0.01 & 0.11 & -0.04 \\ \dots & 0.55 & 0.44 \\ \dots & \dots & -0.88 \end{pmatrix} \qquad \hat{f}_2 \approx \begin{pmatrix} 0.006 & -0.008 & -0.06 \\ \dots & -0.004 & 0.01 \\ \dots & \dots & 1.44 \end{pmatrix}$$

$$f_3 \approx \begin{pmatrix} 0.02 & 0.07 & -0.02 \\ \dots & 0.61 & 0.30 \\ \dots & \dots & 1.58 \end{pmatrix} \qquad \hat{f}_3 \approx \begin{pmatrix} -0.02 & 0.03 & -0.08 \\ \dots & -0.06 & 0.15 \\ \dots & \dots & -1.03 \end{pmatrix}$$

$$f_4 \approx \begin{pmatrix} 0.01 & 0.08 & 0.08 \\ \dots & 0.60 & 0.25 \\ \dots & \dots & -0.19 \end{pmatrix} \qquad \hat{f}_4 \approx \begin{pmatrix} -0.01 & 0.02 & -0.18 \\ \dots & -0.05 & 0.20 \\ \dots & \dots & 0.74 \end{pmatrix}$$

(Rounding off to leading or leading + subleading digit)

Conclusions

- In LR - symmetric models (minimal LR, Pati-Salam, $SO(10)$. E_6,...) one naturally has type I + II seesaw mechanism of neutrino mass generation

- Discrete LR symmetry (parity) leads to a relation between type I and type II contributions to m_ν, which results in a duality property of the seesaw formula: $f \Longleftrightarrow m_\nu/v_L - f$

- For given y, there are 2^n (8 for three lepton generations) matrices f which result in exactly the same mass matrix of light neutrinos m_ν

- A simple analytic method developed for solving the seesaw nonlinear matrix equation. Allows bottom-up reconstraction of the Yukawa coupling matrix f of heavy RH neutrinos

- The results may be used for neutrino mass model building and studies of baryogenesis via leptogenesis. Allow to analytically explore the interplay of type I and II contributions to m_ν

Backup slides

RGE issues

The discrete LR (parity) symmetry of the underlying theory must be broken at some scale v_{LR} $\Rightarrow$ renormalization group evolution below this scale may result in a violation of conditions

$$f_L = f_R \equiv f\,, \qquad\qquad y = y^T$$

at lower energies. Corrections to matrix elements of y and f depend logarithmically on ratios of masses of RH neutrinos and Higgs triplets; are suppressed by loop factors and possibly by small couplings. Numerically checked: If LR-violating corrections to the matr. elements are of the order of percent, reconstruction of the matrix f remains accurate at a percent level.

The stability may be lost if small matrix elements receive corrections proportional to the large elements $\Rightarrow$ a dedicated study necessary

Three lepton generations

Once again go to the basis where $y = diag(y_1,\ y_2,\ y_3)$.

$$f^{-1} = \frac{1}{F}\begin{pmatrix} f_{22}f_{33} - f_{23}^2 & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

System of 6 coupled 4th order equations:

$$x\,F\,(m_{11} - f_{11}) = y_1^2\,(f_{22}f_{33} - f_{23}^2)$$

$$\cdots\cdots\cdots\cdots$$

Simple rescaling would not do! Use similar equations for duals:

$$x\,\hat{F}\,(m_{11} - \hat{f}_{11}) \equiv x\,\hat{F}\,f_{11} = y_1^2\,(\hat{f}_{22}\hat{f}_{33} - \hat{f}_{23}^2)$$

and $(\hat{f}_{22}\hat{f}_{33} - \hat{f}_{23}^2) = (m_{22} - f_{22})(m_{33} - f_{33}) - (m_{23} - f_{23})^2$

Allows to express quadratic in f terms through linear and f-indep. terms and $\hat{F}$

Resulting system:

$$x\,(F+\hat{F})\,f_{11} \;=\; y_1^2\,[(m_{22}\,m_{33} - m_{23}^2) - (m_{22}f_{33} + m_{33}f_{22} - 2m_{23}\,f_{23})] + x\,F\,m_{11}$$

$$\cdots\cdots\cdots\cdots$$

Now one can rescale:

$$f_{ij} \;=\; \lambda^{1/3}\,f'_{ij}\,, \quad m_{ij} \;=\; \lambda^{1/3}\,m'_{ij}\,, \quad y_{ij} \;=\; \lambda^{1/3}\,y'_{ij}\,,$$

Fix λ by requiring $F' \;\equiv\; \det f' \;=\; 1$
Determinant condition: $x^3 F\hat{F} \;=\; -\,y_1^2\,y_2^2\,y_3^2 \quad\Rightarrow$

$$x^3\,\hat{F}' \;=\; -\,(y'_1\,y'_2\,y'_3)^2$$

System of 6 linear eqs. for f'_{ij} – easily solved. Substitution into $F' \;=\; 1$ ($F \;=\; \lambda$) $\quad\Rightarrow\quad$ 8th order characteristic eq. for λ. Yields 4 pairs of dual solutions for the matrix f.

3 generation case – more details

Equations for the matrix elements of f and its dual $\hat{f}$:

$$xF(f_{ij} - m_{ij}) \quad = \quad y_i y_j F_{ij} \ , \qquad (1)$$

$$x\hat{F}(\hat{f}_{ij} - m_{ij}) \quad = \quad -x\hat{F} f_{ij} = y_i y_j \hat{F}_{ij} \ , \qquad (2)$$

Here: $F \equiv \det f$, $\hat{F} \equiv \det \hat{f}$ and

$$F_{ij} \equiv \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} f_{km} f_{ln} \ ,$$

$$\hat{F}_{ij} \equiv \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} \hat{f}_{km} \hat{f}_{ln} = M_{ij} - T_{ij} + F_{ij} \ ,$$

$$M_{ij} \equiv \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} m_{km} m_{ln} \ , \quad T_{ij} \equiv \epsilon_{ikl}\epsilon_{jmn} f_{km} m_{ln} \ .$$

A system of 6 coupled quartic equations for f_{ij}. RH sides are quadratic rather than linear in f_{ij} $\Rightarrow$ a simple rescaling would not linearize the system.
<u>But:</u> using the dual system of eqs. gives for these RH sides

$$y_i y_j F_{ij} = -x\hat{F} f_{ij} + y_i y_j (T_{ij} - M_{ij})$$

linearized by a rescaling as before, except that the scaling factor is now $\lambda^{1/3}$

Linearization: rescaling by a factor $\lambda^{1/3}$

Determinant condition: $x^3 F\hat{F} = -y_1^2\, y_2^2\, y_3^2$. Fix λ by requiring $F'(\lambda) = 1$;
$\Rightarrow \quad \hat{F}' = -(y_1' y_2' y_3')^2/x^3$. The linearized system:

$$[x^3 - (y_1' y_2' y_3')^2] f_{ij}' - x^3 m_{ij}' = x^2 y_i' y_j' (T_{ij}' - M_{ij}')\,.$$

Simplified case: $y_1 \to 0$ (physically well motivated). The result:

$$f_{11} = m_{ee}\,, \qquad f_{12} = m_{e\mu}\,, \qquad f_{13} = m_{e\tau}\,,$$

$$f_{23} = \left(m_{\mu\tau} + \frac{y_2 y_3 m_{e\mu} m_{e\tau}}{x\lambda}\right)/d_2\,,$$

$$f_{22} = \left[m_{\mu\mu} + \frac{y_2^2}{x\lambda}\left(M_{22} - \frac{y_3^2 m_{ee} m_{e\mu}^2}{x\lambda}\right)\right]/d_1\,,$$

$$f_{33} = \left[m_{\tau\tau} + \frac{y_3^2}{x\lambda}\left(M_{33} - \frac{y_2^2 m_{ee} m_{e\tau}^2}{x\lambda}\right)\right]/d_1\,,$$

Here:

$$d_1 = 1 - \frac{y_2^2 y_3^2 m_{ee}^2}{(x\lambda)^2}\,, \qquad d_2 = 1 + \frac{y_2 y_3 m_{ee}}{x\lambda}\,.$$

Characteristic equation for λ:

$$\lambda^4 \left\{[(x\lambda)^2 - m_{ee}^2 y_2^2 y_3^2]^2 - x\left[\det m (x\lambda - m_{ee} y_2 y_3)^2 x\lambda \right.\right.$$
$$\left.\left. + (M_{22} y_2 + M_{33} y_3)^2 (x\lambda)^2\right]\right\} = 0\,.$$

If a solution λ of general charact. eq. $\neq 0$ in the limit $y_1 \to 0$, then $\hat{\lambda} = -y_1^2 y_2^2 y_3^2/(x^3\lambda) \to 0$ $\Rightarrow$ determinant of $\hat{f}$ vanishes. $\Rightarrow$ The dual of any solution that is finite for $y_1 \to 0$ becomes singular and must be discarded.

$\Rightarrow$ For $y_1 \to 0$ there are only 4 (rather than 8) solutions with no duals. The corresp. values of λ are zeros of the factor in curly brackets (which is quartic in λ).

A strong connection with the pure 2-g case

– A different duality among the 4 remaining solutions:

If λ satisfies the charact. equation, so does

$$\tilde{\lambda} \equiv y_2^2 y_3^2 m_{ee}^2/(x^2\lambda)\,,$$

and it corresponds to

$$\tilde{f} \equiv \tilde{m} - f\,,$$

where

$$\tilde{m}_{\alpha\beta} = m_{\alpha\beta} + m_{e\alpha} m_{e\beta}/m_{ee}\,.$$

There are two pairs of such solutions.