AVOIDING THE RENORMALON

IN THE DETERMINATION

OF THE HEAVY QUARK MASSES

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Introduction

Understand the renormalon within an effective field theory/factorization formalism.

Fix the parameters of the Standard Model. Search for weakly sensitive to long distance physics observables. We want to avoid spurious dependence on the renormalon.

Matching coefficients suffer from renormalon ambiguities that cancel with the ones of the matrix elements in effective field theory calculations.

$$c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

Its Borel transform would be

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^{\infty} \mathrm{d}t\, e^{-t/\alpha_s} B[c](t).$$

The ambiguity in the matching coefficient reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where a is a positive number, it sets up the maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation, which is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*},$$

where $n^* \sim \frac{|a|}{\alpha_s}$. Moreover, the fact that a is positive means that, even after Borel resummation, c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{QCD})^{\frac{a\beta_0}{2\pi}} .$$

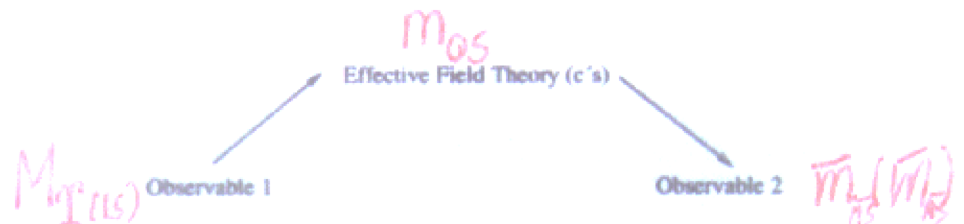


Figure 1: *Symbolic relation between observables through the determination of the matching coefficients of the effective field theory.*

Examples (only true if the perturbative piece can be computed with such precision)

$$\delta_{np} M_B \sim \Lambda_{QCD}, \delta_{np}\Gamma(B \to X_u l\nu) \sim G_F^2 m_{\rm OS}^3 \Lambda_{QCD}^2,$$

$$\delta_{np} M_{\Upsilon(1S)} \sim m_{\rm OS} \frac{\Lambda_{QCD}^4}{(m_{\rm OS}\alpha_s)^4}. \quad (m\alpha^2 >> \Lambda_{QCD})$$

Problem. The OS mass suffers from renormalon ambiguities.

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} \sim \Lambda_{QCD}.$$

Beneke & Braun; Bigi et. al.

Therefore, from perturbation theory

$$\delta_{np}^{(\rm pert.)} M_B \sim \Lambda_{QCD}, \delta_{np}^{(\rm pert.)}\Gamma(B \to X_u l\nu) \sim G_F^2 m_{\rm OS}^4 \Lambda_{QCD},$$

$$\delta_{np}^{(\rm pert.)} M_{\Upsilon(1S)} \sim \Lambda_{QCD}.$$

Proposal: to subtract the renormalon from the matching coefficients. We need to compute them first.

OS mass

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,,$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$\begin{aligned} B[m_{\rm OS}](t(u)) &= N_m \nu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) \\ &+ c_2(1-2u)^2 + \cdots) + (\text{analytic term}), \end{aligned}$$

Next renormalon at $u = 1$.

$$\begin{aligned} r_n &\stackrel{n\to\infty}{=} N_m\,\nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \\ &\times \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right). \end{aligned}$$

Beneke $\quad b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\,b\beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)\,, \qquad \cdots$

Determination of N_m

Lee
$$\begin{aligned} D_m(u) &= \sum_{n=0}^{\infty} D_m^{(n)} u^n = (1-2u)^{1+b} B[m_{\rm OS}](t(u)) \\ &= N_m \nu\, (1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots) \\ &+ (1-2u)^{1+b} (\text{analytic term})\,. \end{aligned}$$

$$N_m \nu = D_m(u = 1/2).$$

$$
\begin{aligned}
N_m &= 0.4244 + 0.1379 + 0.0127 = 0.5750 \quad (n_f = 3) \\
&= 0.4244 + 0.1275 + 0.0004 = 0.5523 \quad (n_f = 4) \\
&= 0.4244 + 0.1199 - 0.0208 = 0.5235 \quad (n_f = 5)
\end{aligned}
$$

Estimates of r_n

$\tilde r_n = r_n/m_{\overline{MS}}$	$\tilde r_0$	$\tilde r_1$	$\tilde r_2$	$\tilde r_3$	$\tilde r_4$
exact ($n_f = 3$)	0.424413	1.04556	3.75086	$---$	$---$
our estimate ($n_f = 3$)	0.617148	0.977493	3.76832	18.6697	118.441
large β_0 ($n_f = 3$)	0.424413	1.42442	3.83641	17.1286	97.5872
exact ($n_f = 4$)	0.424413	0.940051	3.03854	$---$	$---$
our estimate ($n_f = 4$)	0.645181	0.848362	3.03913	13.8151	80.5776
large β_0 ($n_f = 4$)	0.424413	1.31891	3.28911	13.5972	71.7295
exact ($n_f = 5$)	0.424413	0.834538	2.36832	$---$	$---$
our estimate ($n_f = 5$)	0.706913	0.713994	2.36440	9.73117	51.5952
large β_0 ($n_f = 5$)	0.424413	1.21339	2.78390	10.5880	51.3865

Table 1: *Values of r_n for $\nu = m_{\overline{MS}}$. Either the exact result (when available), our estimate, or the estimate using the large β_0 approximation.*

$\tilde r_n = r_n/m_{\overline{MS}}$	$\tilde r_0$	$\tilde r_1$	$\tilde r_2$	$\tilde r_3$	$\tilde r_4$
$O(1/n)$ ($n_f = 3$)	-0.164	-0.046	-0.027	-0.019	-0.015
$O(1/n^2)$ ($n_f = 3$)	0.237	-0.103	-0.017	-0.007	-0.004
$O(1/n)$ ($n_f = 4$)	-0.105	-0.028	-0.016	-0.012	-0.009
$O(1/n^2)$ ($n_f = 4$)	0.274	-0.126	-0.020	-0.008	-0.004
$O(1/n)$ ($n_f = 5$)	0.024	0.006	0.003	0.002	0.002
$O(1/n^2)$ ($n_f = 5$)	0.326	-0.165	-0.023	-0.009	-0.005

Table 2: *$O(1/n)$ corrections (normalized with respect the leading solution) of our r_n estimates for different number of light fermions.*

$$
c_1(n_f = 0) \simeq -0.215, \quad c_2(n_f = 0) \simeq 0.185
$$

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1+(2u-1)\ln\frac{\nu}{m}+\cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u-1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

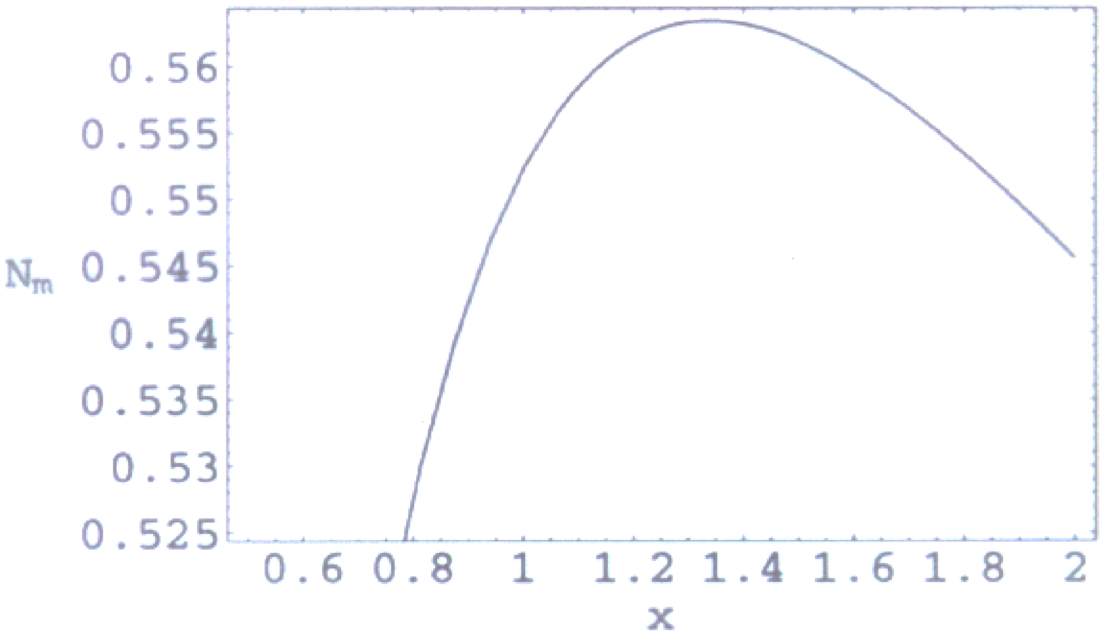


Figure 2: $x \equiv \frac{\nu}{m_{\overline{MS}}}$ *dependence of* N_m *for* $n_f = 4$.

The static potential

$$V_s^{(0)}(r;\nu_{us}) = \sum_{n=0}^{\infty} V_{s,n}^{(0)} \alpha_s^{n+1},$$

$2m_{\rm OS} + V_s^{(0)}$ (not $2m_{\rm OS} + V_o^{(0)}$) can be understood as an observable up to $O(r^2\Lambda_{QCD}^3, \Lambda_{QCD}^2/m)$ renormalon (and/or non-perturbative) contributions. We can use our knowledge of the asymptotic behavior of $m_{\rm OS}$.

$$V_{s,n}^{(0)} \stackrel{n\to\infty}{=} N_V\,\nu\left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}$$
$$\times\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$$2N_m + N_V = 0$$

$$\begin{aligned} D_V(u) &= \sum_{n=0}^{\infty} D_V^{(n)} u^n = (1-2u)^{1+b} B[V_s^{(0)}](t(u)) \\ &= N_V\nu\left(1+c_1(1-2u)+c_2(1-2u)^2+\cdots\right) \\ &+ (1-2u)^{1+b}(\text{analytic term}). \end{aligned}$$

Next (IR) renormalon at $u = 3/2$.

$$\begin{aligned} N_V &= -1.333 + 0.572 - 0.345 = -1.107 \quad (n_f = 3) \\ &= -1.333 + 0.585 - 0.329 = -1.077 \quad (n_f = 4) \\ &= -1.333 + 0.587 - 0.295 = -1.042 \quad (n_f = 5)\,. \end{aligned}$$

$$2\frac{2N_m + N_V}{2N_m - N_V} = \begin{cases} 0.038 & , n_f = 3 \\ 0.025 & , n_f = 4 \\ 0.005 & , n_f = 5. \end{cases}$$

Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.

The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}} \times \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right),$$

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}.$$

(RS') (n=1)

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1}(\nu),$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.

Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m \nu_f \left(1 + c_1 + c_2 + \cdots\right).$$

Check of convergence improvement

Masses	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$m_{\rm OS}$	401	199	144	147	5 102
$m_{\rm RS}$	111	50	17	7	4 395
$m_{\rm RS'}$	401	114	38	15	4.778
$m_{\rm PS}$	210	80	42	– – –	4 542
$m_{1S}^{(\rm static)}$	102	50	19	8	4 389
$m_{\rm RS}$	256	95	40	21	4 622
$m_{\rm RS'}$	401	157	74	41	4.882
$m_{\rm PS}$	306	120	67	– – –	4.703
$m_{1S}^{(\rm static)}$	251	94	41	22	4 619

Table 4: *Contributions at various orders in α_s for different mass definitions for the bottom quark case, either with $\nu_f = 1/r = 2$ GeV (middle panel) or with $\nu_f = 1/r = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 1 has been used. The other parameters have been fixed to the values $m_{\overline{\rm MS}}(m_{\overline{\rm MS}}) = 4.21$ GeV, $\nu = m_{\overline{\rm MS}}(m_{\overline{\rm MS}})$ and $n_f = 4$.*

$$m_{1S}^{(\rm static)} \equiv m_{\rm OS} + \frac{V(r)}{2} = m_{\overline{\rm MS}} + \left(r_0 - \frac{C_f}{2r}\right)\alpha_s + \cdots .$$

HQET

$$\mathcal{L} = \bar{h}\left(iD_0 - \delta m_{\rm RS}\right)h + O\left(\frac{1}{m_{RS}}\right),$$

where $\delta m_{\rm RS} = m_{\rm OS} - m_{\rm RS}$ and similarly for the NRQCD Lagrangian.

Weakly sensitive to long distance physics observable

$$O\left(\frac{\Lambda^2}{m}\right) \quad \langle M_B\rangle - \langle M_D\rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1\left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}}\right)$$

$$O(\Lambda) \quad O(\Lambda) \quad + O(1/m_{\rm RS}^2).$$

pNRQCD. If $\Lambda_{QCD} \ll m\alpha_s$

$$V^{(0)}_{s,\mathrm{RS(RS')}}(\nu_f) = V^{(0)}_s + 2\delta m_{\mathrm{RS(RS')}} ,$$

Check of convergence improvement

Potentials	$O(\alpha_s)$	$O(\alpha_s^2)$	$O(\alpha_s^3)$	$O(\alpha_s^4)$	total
$V_s^{(0)}$	−910	−306	−302	−383	−1 902
$V_{s,\mathrm{RS}}^{(0)}$	−205	3	−2	−3	−208
$V_{s,\mathrm{RS'}}^{(0)}$	−910	−54	−14	−6	−984
$V_{s,\mathrm{PS}}^{(0)}$	−446	−42	−25	— — —	−513
$V_{s,\mathrm{RS}}^{(0)}$	−558	−63	−41	−26	−687
$V_{s,\mathrm{RS'}}^{(0)}$	−910	−180	−95	−54	−1 239
$V_{s,\mathrm{PS}}^{(0)}$	−678	−116	−75	— — —	−869

Table 5: *Contributions at various orders in α_s for different singlet static potential definitions for some typical scales in the Υ system, either with $\nu_f = 2$ GeV (middle panel) or with $\nu_f = 1$ GeV (lower panel). The results are displayed in MeV. For the $O(\alpha_s^4)$ results, the estimate from Table 3 has been used. The other parameters have been fixed to the values $\nu = 1/r = 2.5$ GeV and $n_f = 4$.*

pNRQCD Lagrangian

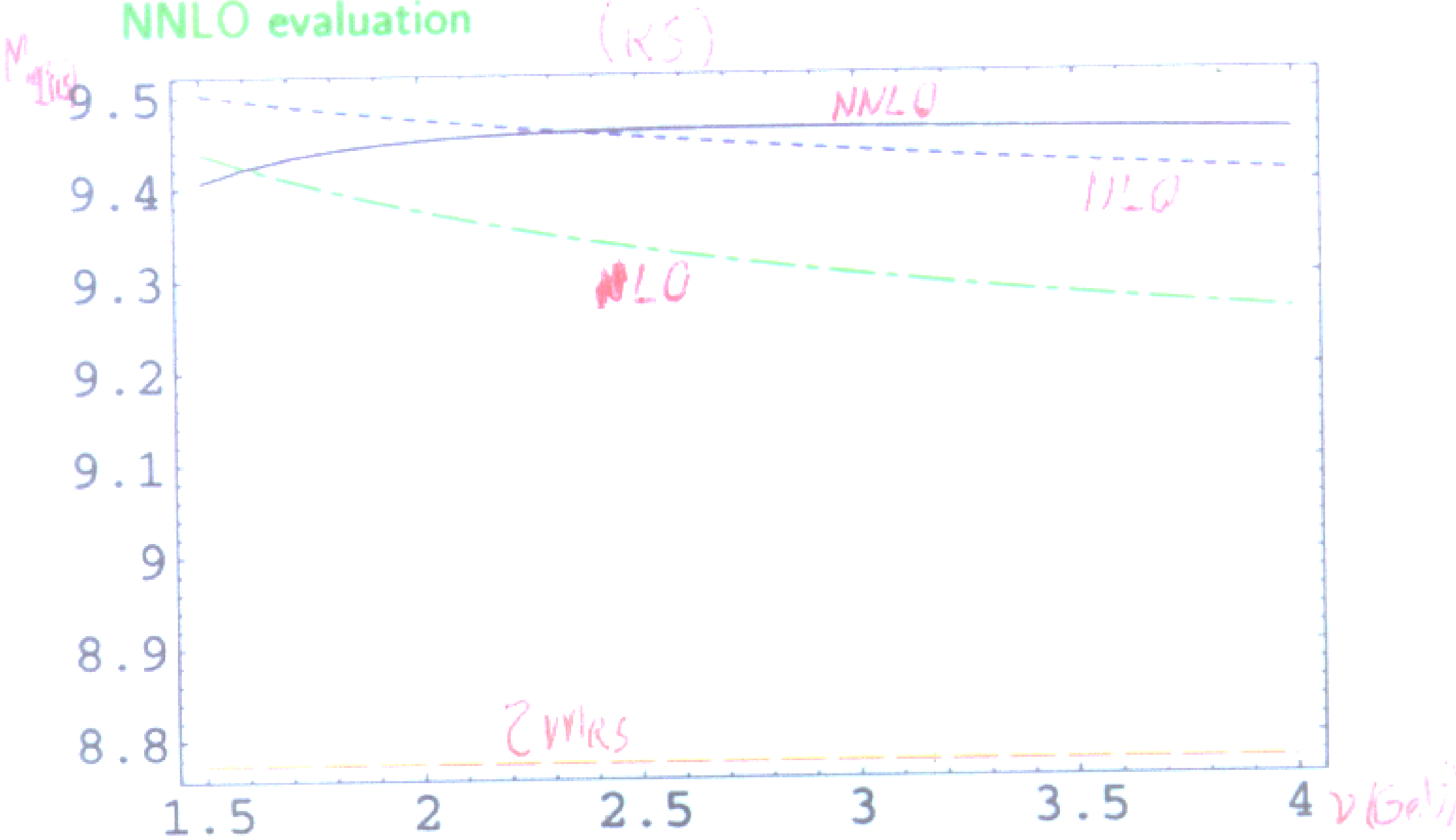
$$\begin{aligned}
\mathcal{L}^{(0)} &= \mathrm{Tr}\Big\{S^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{s,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) S \\
&+ O^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \sum_n \frac{V^{(n)}_{o,\mathrm{RS}}(\mathbf{x})}{m^n_{\mathrm{RS}}}\right) O\Big\} \\
&+ gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, S + S^\dagger \mathbf{x}\cdot\mathbf{E}\, O\right\} \\
&+ g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger \mathbf{x}\cdot\mathbf{E}\, O + O^\dagger O \mathbf{x}\cdot\mathbf{E}\right\} ,
\end{aligned}$$

Weakly sensitive to long distance physics observable

$$M_{nlj} = 2m_{\mathrm{RS}} + \sum_{m=2}^{\infty} A^{m,\mathrm{RS}}_{nlj}(\nu_{us})\alpha_s^m + \underbrace{\delta M^{\mathrm{US}}_{nlj}(\nu_{us})}_{O\left(\frac{\Lambda_c^3}{(m\alpha)^2}\right) \text{ or } O\left(\frac{\Lambda^4}{m^3\alpha^4}\right)} .$$

Bottom $\overline{\mathrm{MS}}$ quark mass determination

NNLO evaluation



Dependence on the parameters for the RS scheme: $\nu = 2.5^{+1.5}_{-1}$ GeV, $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.118 \pm 0.003$ and $N_m = 0.552 \pm 0.0552$

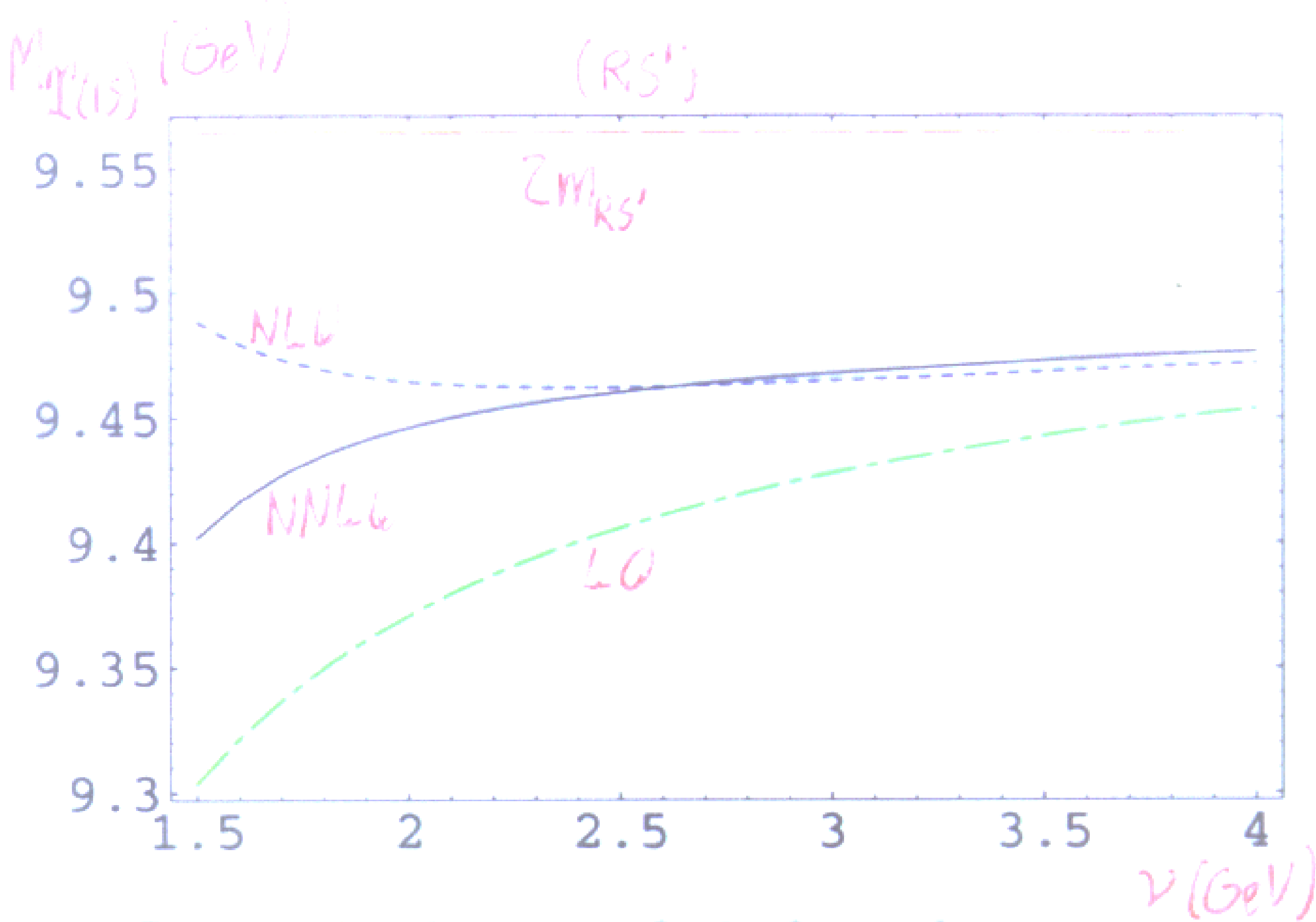
$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4\,387^{+2}_{+28}(\nu)^{-5}_{+7}(\nu_f)^{-16}_{+16}(\alpha_s)^{-68}_{+68}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,203^{+2}_{+25}(\nu)^{-5}_{+6}(\nu_f)^{-28}_{+27}(\alpha_s)^{-10}_{+10}(N_m)\ \mathrm{MeV}.$$

Convergence. In the RS scheme

$$M_{\Upsilon(1S)} = 8\,774 + 559 + 120 + 7\ \mathrm{MeV}.$$

NNLO(st. pot.) $\sim +62$ MeV. NNLO(rel.) ~ -55 MeV.



For the RS' scheme, we obtain the result

$$m_{b,\mathrm{RS}'}(2\ \mathrm{GeV}) = 4\,782^{-08}_{+31}(\nu)^{-7}_{+3}(\nu_f)^{+15}_{-12}(\alpha_s)^{-28}_{+28}(N_m)\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{-08}_{+28}(\nu)^{-6}_{+3}(\nu_f)^{-25}_{+25}(\alpha_s)^{-9}_{+9}(N_m)\ \mathrm{MeV}.$$

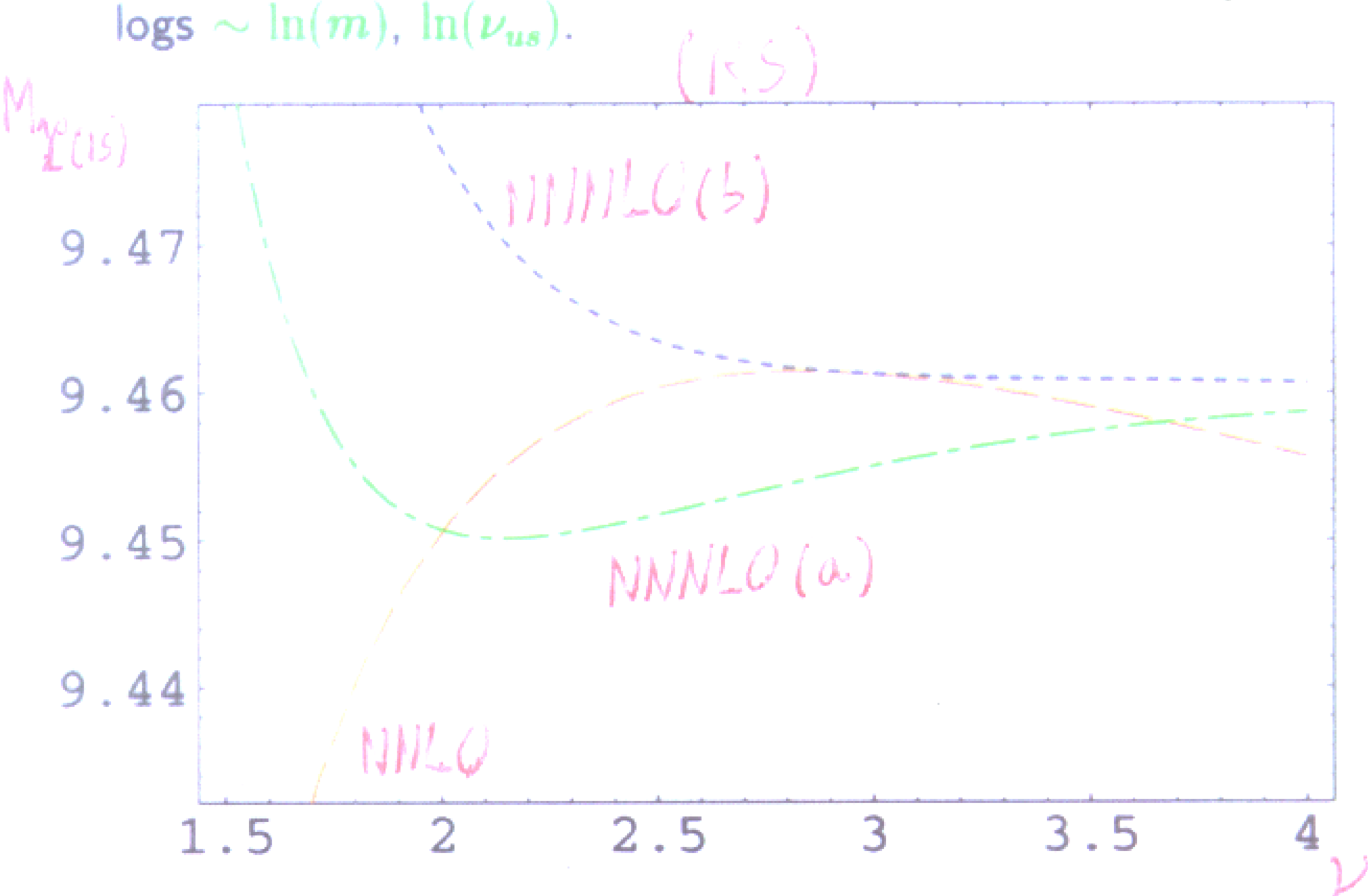
Convergence. In the RS' scheme

$$M_{\Upsilon(1S)} = 9\,564 - 158 + 56 - 2\ \mathrm{MeV}\,.$$

NNLO(st. pot.) $\sim +45$ MeV. NNLO(rel.) ~ -47 MeV.

NNNLO evaluation. Exact scale dependence + large β_0 estimate for the log-independent piece of $A^{5,\mathrm{OS}}_{101}$.

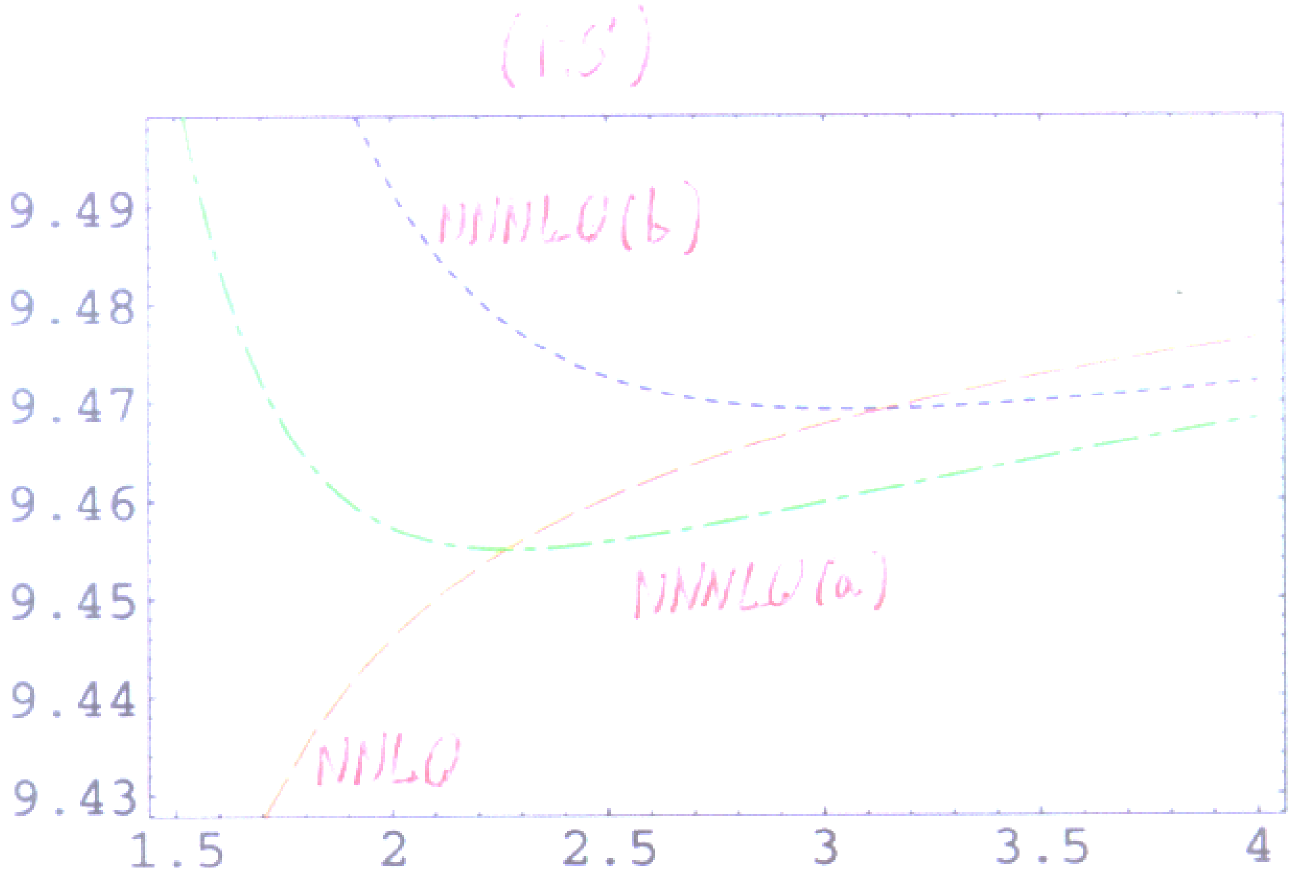
Two kind of logs: (a) $\sim \ln(m\alpha_s/\nu)$ and (b) hard/ultrasoft logs $\sim \ln(m)$, $\ln(\nu_{us})$.



Strong scale dependence at small ν (renormalization group improvement needed?)

NNNLO contribution seems to be under control.

Error $\sim \pm 50$ MeV (including charm mass effects).


(1S)
NNNLO(b)
MNNLO(a)
NNLO
9.49
9.48
9.47
9.46
9.45
9.44
9.43
1.5
2
2.5
3
3.5
4

What about US effects?

$$\delta M_{nlj}^{\mathrm{US}}(\nu_{us}) \simeq \frac{T_F}{3N_c}\int_0^\infty dt \langle n,l|\mathbf{r}e^{-t(H_o^{\mathrm{RS}}-E_n^{\mathrm{RS}})}\mathbf{r}|n,l\rangle$$
$$\times\langle g\mathbf{E}^a(t)\phi(t,0)_{ab}^{\mathrm{adj}} g\mathbf{E}^b(0)\rangle(\nu_{us}),$$

where $H_o^{\mathrm{RS}} \equiv \dfrac{\mathbf{p}^2}{m_{\mathrm{RS}}} + \dfrac{1}{2N_c}\dfrac{\alpha_s}{r}$ and $E_n^{\mathrm{RS}} \equiv -m_{\mathrm{RS}} C_f^2 \alpha_s^2/(4n^2)$.

Situation $\Lambda_{QCD} \ll m\alpha_s^2$

$$\delta M_{nl}^{\mathrm{US}}(\nu_{us}) = \delta M_{nl}^{\mathrm{US,\,pert.}} + \delta M_{nl}^{\mathrm{US,\,no-pert.}},$$

$$\delta M_{nl}^{\mathrm{US,\,pert.}} \sim m_{\mathrm{RS}}\alpha_s^5 \ln\frac{\nu_{us}}{m_{\mathrm{RS}}\alpha_s^2},$$

$$\delta M_{nl}^{\mathrm{US,\,no-pert.}} = \sum_{n=0}^{\infty} C_n O_n\,,$$

where $C_n \sim 1/(m_{\mathrm{RS}}^{3+2n}\alpha_s^{4+4n})$ and $O_n \sim \Lambda_{QCD}^{4+2n}$.

Problems: (1) Dependence on $\alpha_s(m\alpha_s^2)$. $\langle H_o^{RS} - E_1^{RS}\rangle_{10} \sim$ 360 MeV (up to numerical factors). (2) Convergence of the OPE (?).

$$C_0O_0 + C_1O_1 = 144 - 143\ \mathrm{MeV}\,. \quad (\nu \simeq 2.5\ \mathrm{GeV})$$

The situation improves by lowering the scale ν (97 − 66 MeV for $\nu = 2$ GeV and 53 − 21 MeV for $\nu = 1.5$ GeV) and it also depends on the poorly known values of the condensates.

Situation $\Lambda_{QCD} \sim m\alpha_s^2$. Non-local condensate. Basically unknown. Formally NNLO effect ($\sim$ 50 MeV)

Error $\sim \pm 100$ MeV.

Checking the stochastic vacuum model

$$\langle gF^a_{\mu\nu}(x)\phi(x,0)^{\rm adj}_{ab}gF^b_{\rho\sigma}(0)\rangle$$
$$= (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\left[\mathcal{D}(x^2) + \mathcal{D}_1(x^2)\right]$$
$$+(x_\nu x_\sigma\delta_{\mu\rho} - x_\nu x_\rho\delta_{\mu\sigma} - x_\mu x_\sigma\delta_{\nu\rho} + x_\mu x_\rho\delta_{\nu\sigma})\frac{\partial\mathcal{D}_1(x^2)}{\partial x^2}.$$

For the gauge string a straight line is understood and $\mathcal{D}$ and $\mathcal{D}_1$ are invariant functions of x^2. In our case the following combination appears

$$\langle g\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab}g\mathbf{E}^b(0)\rangle = 3\left(\mathcal{D}(t^2) + \mathcal{D}_1(t^2) + t^2\frac{\partial\mathcal{D}_1(t^2)}{\partial t^2}\right).$$

In the situation $mv \gg \Lambda_{QCD} \gg mv^2$ the leading non-perturbative effects can be parameterized in terms of a potential term as follows

$$\delta V = r^2\frac{T_F}{3N_c}\int_0^\infty dt\langle g\mathbf{E}^a(t)\phi(t,0)^{\rm adj}_{ab}g\mathbf{E}^b(0)\rangle(\nu_{us}).$$

The stochastic vacuum model gives the following result:

$$\delta V = r^2\frac{T_F}{N_c}\int_0^\infty dt\left(\mathcal{D}(t^2) + \frac{1}{2}\mathcal{D}_1(t^2)\right).$$

This result does not appear to be able to reproduce the leading logs predicted by perturbation theory.

Final results (RS$= N_m + \nu_f$)

RS scheme

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = 4\,387^{+75}_{-75}(\mathrm{US/N^3LO})^{-16}_{+16}(\alpha_s)^{-73}_{+75}(\mathrm{RS})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,203^{+67}_{-67}(\mathrm{US/N^3LO})^{-28}_{+27}(\alpha_s)^{-15}_{+16}(\mathrm{RS})\ \mathrm{MeV}.$$

RS' scheme

$$m_{b,\mathrm{RS'}}(2\ \mathrm{GeV}) = 4\,782^{+75}_{-75}(\mathrm{US/N^3LO})^{+15}_{-12}(\alpha_s)^{-35}_{+31}(\mathrm{RS'})\ \mathrm{MeV};$$

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,214^{+67}_{-67}(\mathrm{US/N^3LO})^{-25}_{+25}(\alpha_s)^{-15}_{+12}(\mathrm{RS'})\ \mathrm{MeV}.$$

We average the two values obtained for the $\overline{\mathrm{MS}}$ mass. We then obtain (rounding)

$$m_{b,\overline{\mathrm{MS}}}(m_{b,\overline{\mathrm{MS}}}) = 4\,210^{+90}_{-90}(\mathrm{theory})^{-25}_{+25}(\alpha_s)\ \mathrm{MeV}.$$

Charm $\overline{\rm MS}$ quark mass determination

Weakly sensitive to long distance physics observable.

$$O\left(\frac{\Lambda^2}{m}\right) \quad \langle M_B \rangle - \langle M_D \rangle = m_{b,\rm RS} - m_{c,\rm RS} + \lambda_1 \left(\frac{1}{2m_{b,\rm RS}} - \frac{1}{2m_{c,\rm RS}} \right)$$

$$O(\Lambda) \qquad O(\Lambda) \qquad + O(1/m_{\rm RS}^2)\,,$$

RS scheme ($\alpha_s(M_z) = 0.118 \pm 0.003$, $N_m = 0.552 \pm 0.0552$, $\lambda_1 = 0.3 \pm 0.2$ and $m_{b,\overline{\rm MS}} = 4\,203^{+67}_{-67}$ MeV) $(\nu_f = 1\,{\rm GeV})$

$$m_{c,\rm RS}(1\ {\rm GeV}) = 1\,181^{+82}_{-84}(m_{b,\overline{\rm MS}})^{+4}_{-1}(\alpha_s)^{-50}_{+50}(N_m)^{-78}_{+65}(\lambda_1)\ {\rm MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,206^{+66}_{-67}(m_{b,\overline{\rm MS}})^{+1}_{-0}(\alpha_s)^{+11}_{-13}(N_m)^{-62}_{+52}(\lambda_1)\ {\rm MeV},$$

RS' scheme ($m_{b,\overline{\rm MS}} = 4\,214^{+67}_{-67}$ MeV)

$$m_{c,\rm RS'}(1\ {\rm GeV}) = 1\,477^{+79}_{-80}(m_{b,\overline{\rm MS}})^{+34}_{-27}(\alpha_s)^{-19}_{+19}(N_m)^{-54}_{+48}(\lambda_1)\ {\rm MeV};$$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,207^{+65}_{-64}(m_{b,\overline{\rm MS}})^{-7}_{+5}(\alpha_s)^{+13}_{-14}(N_m)^{-43}_{+39}(\lambda_1)\ {\rm MeV}.$$

The perturbative relation between the RS and $\overline{\rm MS}$ charm mass is convergent.

$$m_{c,\rm RS}(1\ {\rm GeV}) = 1\,206 - 53 + 20 + 6 + 3 = 1\,181\ {\rm MeV},$$
$$m_{c,\rm RS'}(1\ {\rm GeV}) = 1\,207 + 205 + 46 + 13 + 6 = 1\,477\ {\rm MeV}\,.$$

Other sources of error. (1) $\pm 40(20)$ MeV error to the RS'(RS) evaluation due to the conversion from the $\overline{\rm MS}$ to the RS'(RS) bottom quark mass. (2) $1/m^2$ terms. $\sim 15(30)$ MeV error to the RS'(RS) evaluation.

Our final result reads

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,210^{+70}_{-70}({\rm theory})^{+65}_{-65}(m_{b,\overline{\rm MS}})^{-45}_{+45}(\lambda_1)\ {\rm MeV}.$$

Determination of $\bar{\Lambda}$

$$\bar{\Lambda}_{\rm RS} = \langle M_B \rangle - m_{b,\rm RS} - \frac{\lambda_1}{2m_{b,\rm RS}} + O(1/m_{b,\rm RS}^2)\,.$$

We obtain (using $m_{b,\overline{\rm MS}} = 4\,210$ **MeV**)

$$\bar{\Lambda}_{\rm RS}(1~\mathrm{GeV}) = 659~\mathrm{MeV}, \qquad \bar{\Lambda}_{\rm RS'}(1~\mathrm{GeV}) = 401~\mathrm{MeV}.$$

We can see that it is crucial to specify the scheme in order to give a meaningful prediction for $\bar{\Lambda}$.

Check. Calculation in the OS scheme + renormalon cancellation ($\nu = m_{c,\overline{\rm MS}}$). $m_{b,\overline{\rm MS}} = 4\,210 \pm 90$

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,254^{+85}_{-84}(m_{b,\overline{\rm MS}})^{+17}_{-12}(\alpha_s)^{-49}_{+45}(\lambda_1)~\mathrm{MeV}\,.$$

$$m_{b,\rm OS} - m_{c,\rm OS} = 2\,956 + 490 - 14 - 32 + 22 = 3\,423~\mathrm{MeV}\,.$$

Potential large logs. With $\nu = 2m_{c,\overline{\rm MS}}$, we obtain

$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,239\mathrm{MeV}$$

and the expansion seems to improve:

$$m_{b,\rm OS} - m_{c,\rm OS} = 2\,971 + 345 + 79 + 19 + 10 = 3\,424~\mathrm{MeV}.$$

Conclusions

- Calculation of the normalization constant of the first renormalon of the pole mass and static potential..

$$N_m = 0.55 \qquad N_V = -1.08\ (n_f = 4)$$

- Estimates of the higher order coefficients of the perturbative series relating the pole mass with the $\overline{\rm MS}$ mass (and the singlet static potential with α_s) have been obtained without relying on the large β_0 approximation.
- Renormalon free working scheme preserving the power counting rules.
- Determination of the $\overline{\rm MS}$ bottom and charm masses.

$$m_{b,\overline{\rm MS}}(m_{b,\overline{\rm MS}}) = 4\,210^{+90}_{-90}(\text{theory})^{-25}_{+25}(\alpha_s)\ \text{MeV},$$
$$m_{c,\overline{\rm MS}}(m_{c,\overline{\rm MS}}) = 1\,210^{+70}_{-70}(\text{theory})^{+65}_{-65}(m_{b,\overline{\rm MS}})^{-45}_{+45}(\lambda_1)\text{MeV}.$$