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Problems and challenges in perturbative QCD

- Higher Order calculations:
  Where are we headed?
- Shower Montacarlo
  The inclusion of higher order effects
- Resummation
  Some open problems in the resummation of soft gluons

Higher order calculations: why bother?

There is a sector of NNLO calculations which is very advance, has made continuous progress for many years, and has reached a remarkable success in QCD tests and in the extraction of α_S

- $R(e^+e^- \to \text{hadrons})$ at $\mathcal{O}(\alpha_S^3)$
  (Gorishny, Kataev, Larin, 88, 91, and
  Surguladze, Samuel (91))

$$\frac{R}{R_0} = 1 + \frac{\alpha_S}{\pi} + 1.409\left(\frac{\alpha_S}{\pi}\right)^2 - 12.805\left(\frac{\alpha_S}{\pi}\right)^3 + \mathcal{O}(\alpha_S^4)$$

- The QCD β function, computed at 4 loops
  (van Ritbergen, Vermaseren, Larin, 1997)

$$\frac{\partial \alpha_S}{\partial \ln \mu^2} = -\frac{\beta_0}{4\pi}\alpha_S^2 - \frac{\beta_1}{(4\pi)^2}\alpha_S^3 - \frac{\beta_2}{(4\pi)^3}\alpha_S^4 - \frac{\beta_3}{(4\pi)^4}\alpha_S^5 + O(\alpha_S^6)$$

$$\beta_0 \approx 11 - 0.66667 n_f\,, \qquad \beta_1 \approx 102 - 12.6667 n_f$$

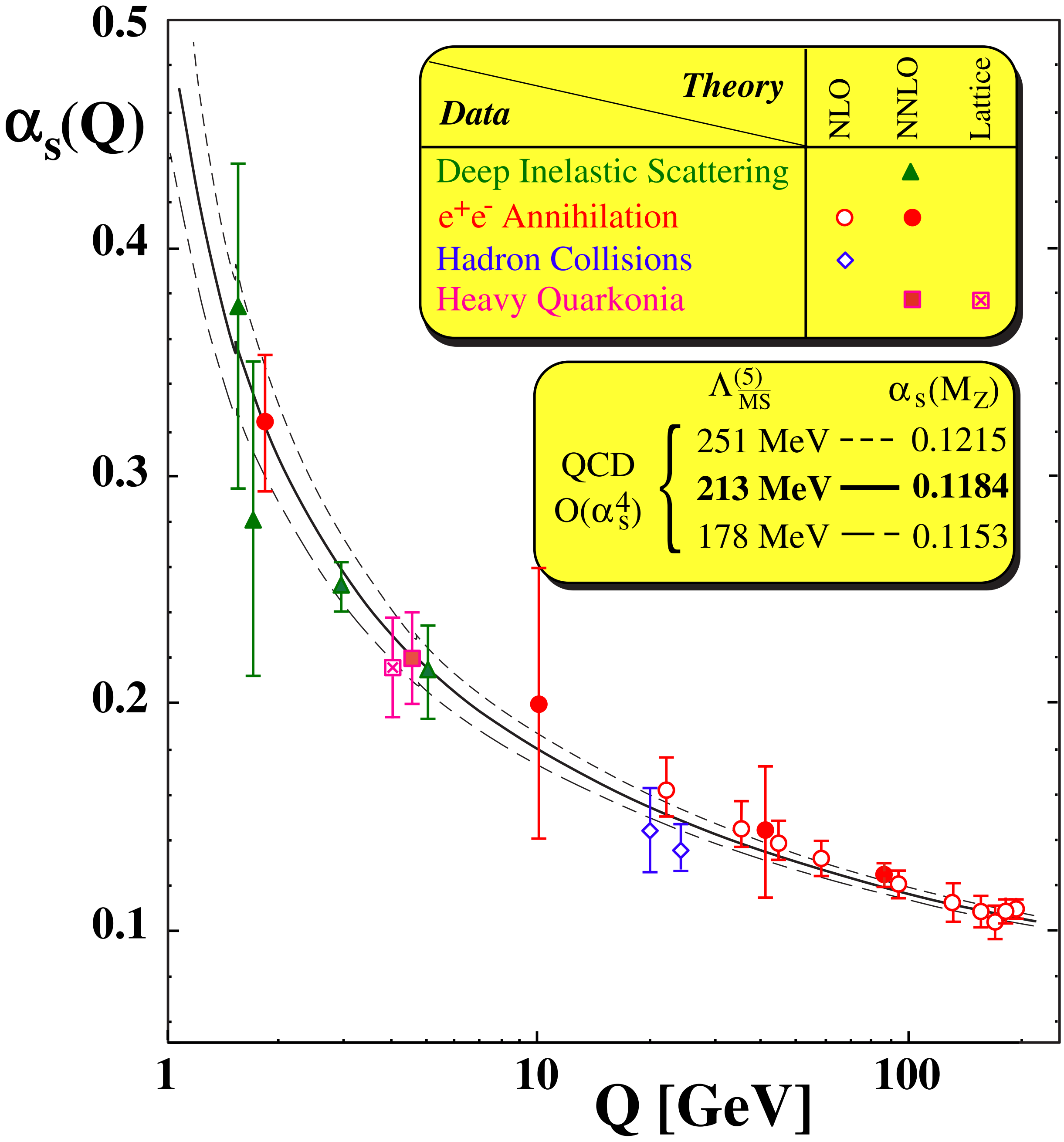
$$\beta_2 \approx 1428.50 - 279.611 n_f + 6.01852 n_f^2$$

$$\beta_3 \approx 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3$$

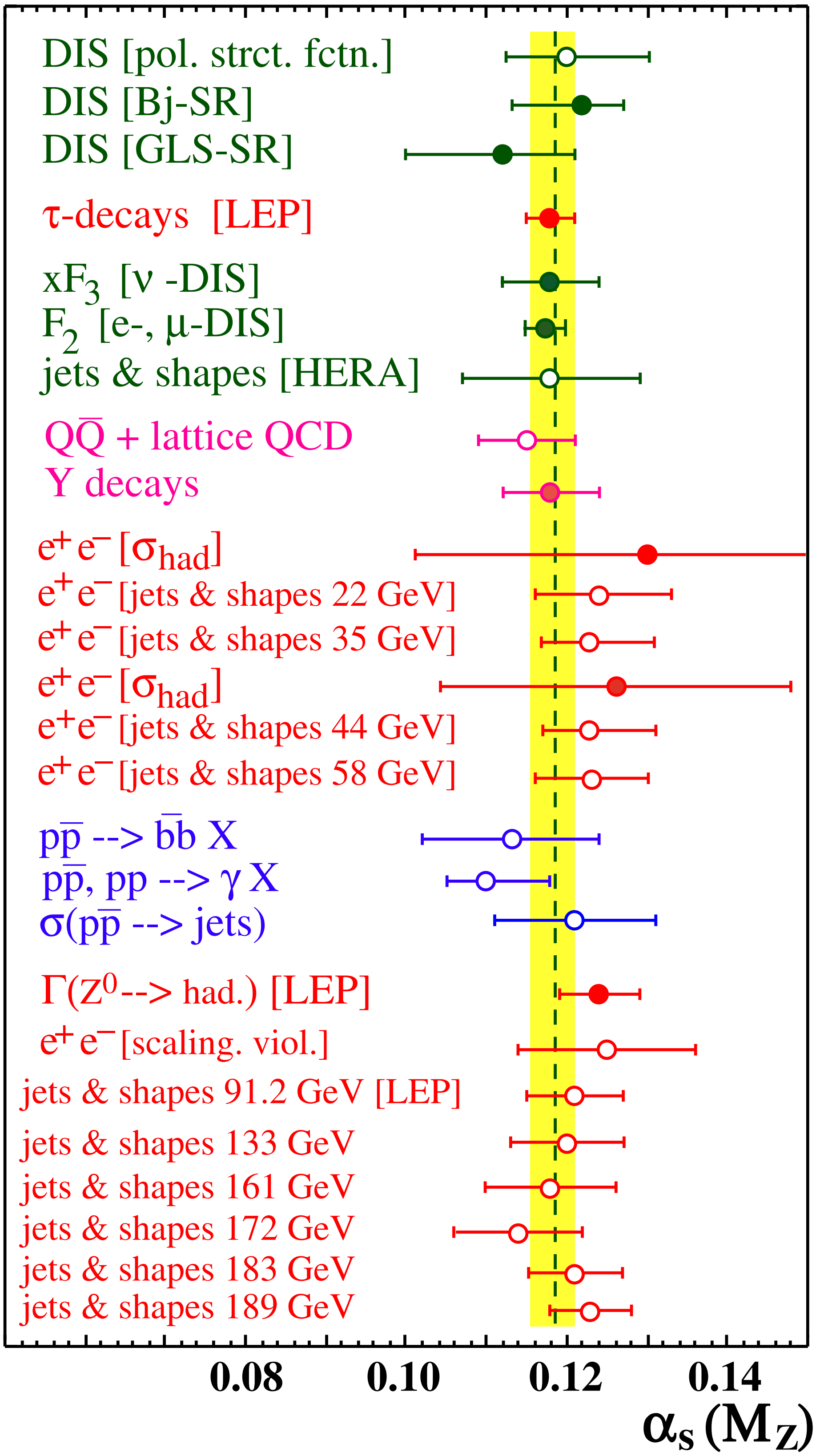
- Up to $N = 12$ singlet, $N = 14$ non-singlet crossing even, and $N = 13$ crossing odd moments of the $\mathcal{O}(\alpha_S^3)$ splitting functions, together with the $\mathcal{O}(\alpha_S^3)$ coefficient functions for DIS
  (Larin, Nogueira, van Ritbergen, Vermaseren; Retey and Vermaseren; Moch and Vermaseren; Zijstra and van Neerven)

All these results are obtainable from propagator type, massless graphs (the MINCER program, Gorishnii, S.A. Larin, L.R. Surguladze, F.V. Tkachov, 1989) Larin, Tkachov, Vermaseren, 1991. Massive graphs can be computed in a m^2/q^2 power expansion (Chetyrkin, Harlander, Kühn and Steinhauser), or in a q^2/m^2 power expansion, from tadpole type massive graphs (Larin, Ritbergen, Vermaseren).

These results have an important impact on α_S determinations. The situation is summarized (S. Bethke, April 2000) in the following plot:



in terms of $\alpha_s(M_Z)$:



At LEP, $\alpha_{\rm s}$ has also been measured most accurately from the Z^0 hadronic width (assume no new physics effects),

$$\alpha_{\rm s}(M_{\rm z}) = 0.123 \pm 0.004 (\text{from } R_l)\,, \quad = 0.118 \pm 0.003 (\text{from } \sigma_l^0)$$

and from τ hadronic decays (assume τ heavy enough):

$$\alpha_{\rm s}(M_{\rm z}) = 0.1181 \pm 0.0007(\text{exp.}) \pm 0.003(\text{theo.}).$$

For τ decays, both ALEPH and OPAL have performed measurements of the vector and axial vector spectral mass distributions from τ decays, allowing several consistency checks.

Consistency of the two determination is very encouraging (theoretically similar, very different scales). Both determination are at the NNLO level.

Other determination with similar accuracy?

DIS: power corrections $\approx 1/Q^2$, some NNLO results available; Kataev, Parente, Sidorov have reanalized CCFR data in NNLO QCD, and obtained

$$\alpha_{\rm s}(M_{\rm z}) = 0.118 \pm 0.002(\text{stat.}) \pm 0.005(\text{sys}) \pm 0.003(\text{theo.})$$

Santiago and Yndurain analyzed F_2 in μ and e DIS from 2.5 to 230 GeV^2 at NNLO, and obtained

$$\alpha_{\rm s}(M_{\rm z}) = 0.1166 \pm 0.0013$$

and from νN scattering

$$\alpha_{\rm s}(M_{\rm z}) = 0.1153 \pm 0.0063$$

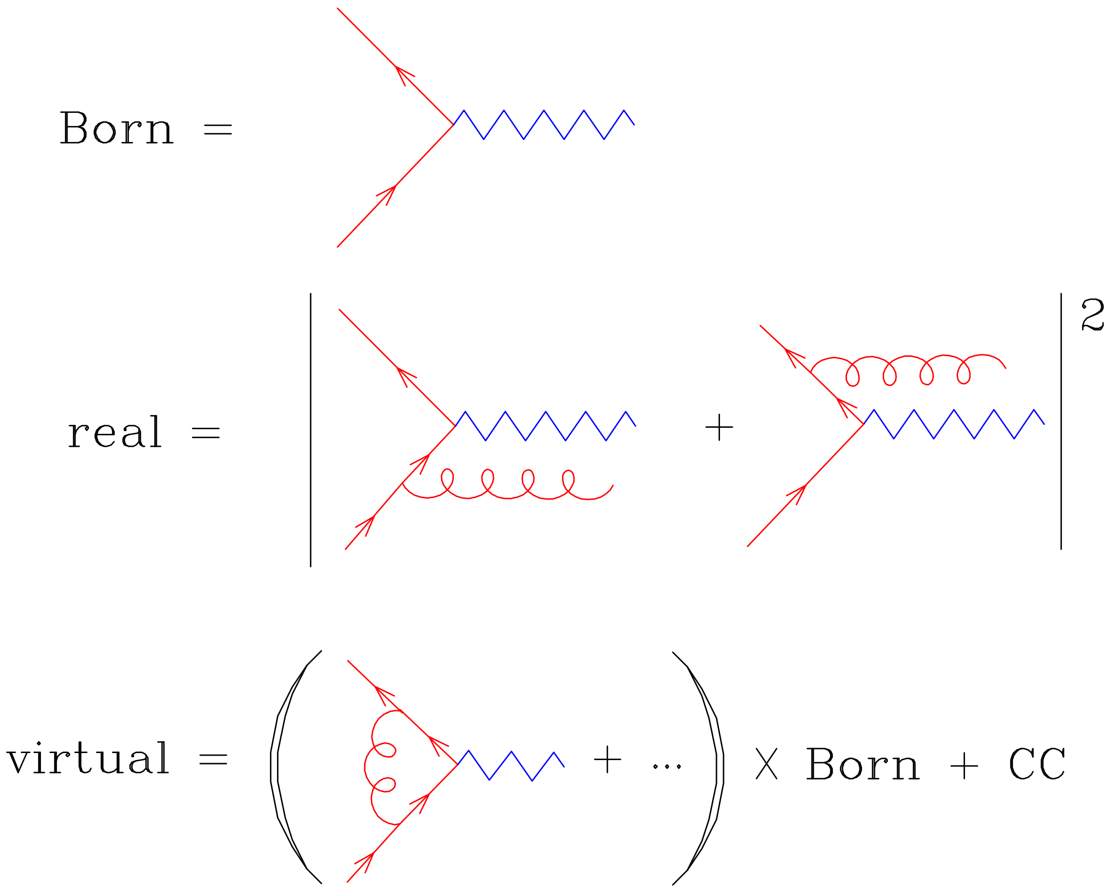
For α_{S} determinations, inclusive hadronic decays and DIS determinations are better.

Power corrections are smaller ($1/Q^2$, $1/Q^4$), and NNLO effects are included. All these determinations are remarkably consistent.

Aside from these cases, the standard QCD calculations for collider physics are done at NLO. NNLO results are available only in the very simple case of Drell-Yan pair production. Some of the most important results:

- Drell-Yan pair production (Altarelli, Ellis, Martinelli; Kubar-Andre, Paige; 1978), also at NNLO (van Neerven, Matsuura, van der Marck, Hamberg; 1991)
- Jet production in e^+e^- annihilation (Ellis, Ross and Terrano; 1980)
- Jet production in hadronic collisions (S. Ellis, Kunszt, Soper; Aversa, Chiappetta, Greco and Guillet; 1990) building on work of Ellis and Sexton, 1986.
- heavy flavour hadroproduction K. Ellis, P.N. 1988, (Mangano, Nason, Ridolfi, 1992)
- Direct photon production (Aurenche, Baier, Fontannaz, Schiff, 1988), with polarized beams (Gordon, Vogelsang, 1996)
- Vector bosons pair production (Frixione, Mele, Ridolfi, P.N.; Owens, Ohnemus; 1991)
- Higgs production Graudenz, Spira, Zerwas, 1993

NLO calculations are today standard, with NLL evolution of the parton densities. Basic structure:



Real and virtual corrections are regularized in $d = 4 - 2\epsilon$ dimensions, and singularities in $1/\epsilon$ and $1/\epsilon^2$ appear in both real and virtual corrections, as a consequence of the collinear and soft divergences in the calculation. Singularities cancel in inclusive quantities; however, the implementation is problematic (4 years between the Ellis-Sexton calculation for the jet matrix elements and the implementation in a phenomenologically useful code).

The implementation of IR cancellations turns the differential cross sections into distributions in the final state kinematical variables. A simple example:
one variable $0 < x < 1$, the cross section (Born +NLO) is

$$\frac{d\sigma}{dx} = \delta(x) + \alpha_S \left(\frac{1}{x}\right)_+$$

can be implemented either by slicing away a small region around $x = 0$ (Slicing method):

$$\frac{d\sigma}{dx} = \delta(x) + \alpha_S \theta(x-\eta)\frac{1}{x} - \alpha_S log\frac{1}{\eta}\delta(x)\,,$$

or subtracting under the integral sign when integrating the cross section agains a physical observable
(Subtraction method):

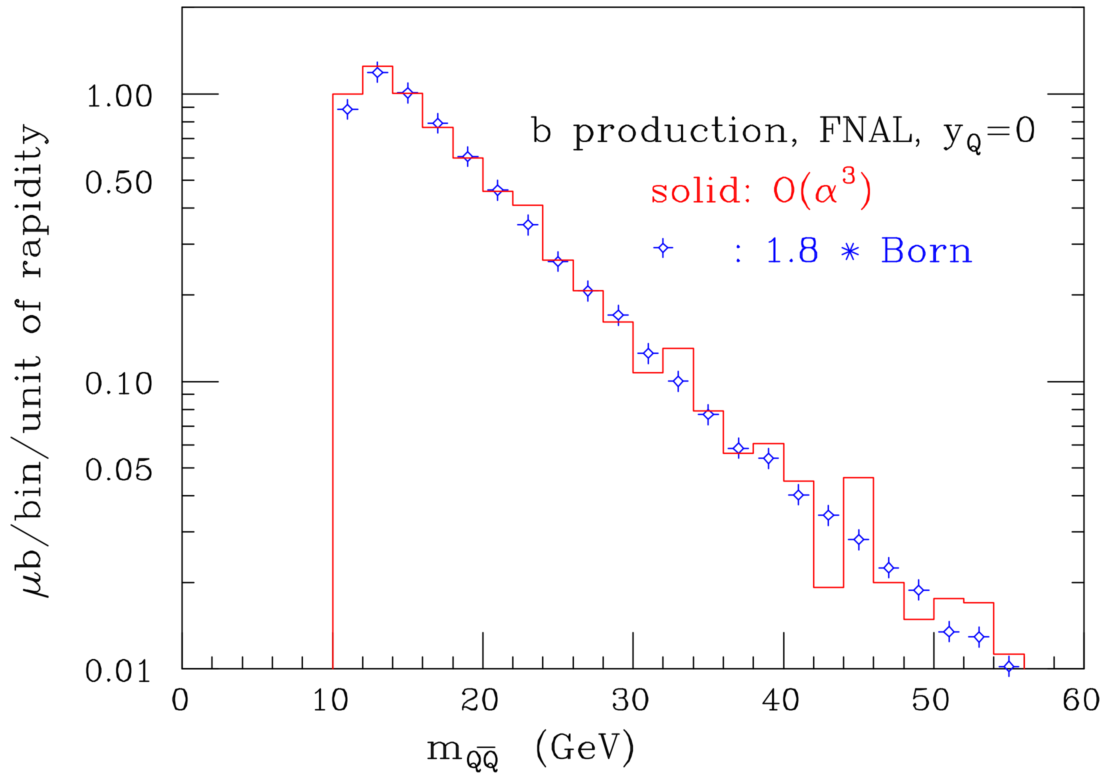
$$\int G(x)\frac{d\sigma}{dx}dx = G(0) + \alpha_S \int \frac{G(x)-G(0)}{x}dx\,.$$

The subtraction method, first introduced by Ellis, Ross and Terrano, has the advantage that it does not need an unphysical cut-off η, and it does not lead to cancellations between large numbers when the cut-off is small. In the program EVENT (Kunszt, P.N., 1990) this method has been implemented as a black box parton montecarlo generator easily usable by experimentalists. The same technique has also been introduced in the context of hadronic collisions in vector boson pair production (Mele, Ridolfi, P.N.) and in jet physics (S.Ellis, Kunszt, Soper).

Parton generator

- Generate one event with $x = 0$ and weight 1 (the born term).
- Generate N random value $0 < \tilde{x} < 1$
- for each random value $\tilde{x}$ generate 2 events
  - one event with $x = \tilde{x}$ and weight $\frac{\alpha_S}{N}\frac{1}{\tilde{x}}$
  - one event with $x = 0$ and weight $-\frac{\alpha_S}{N}\frac{1}{\tilde{x}}$.

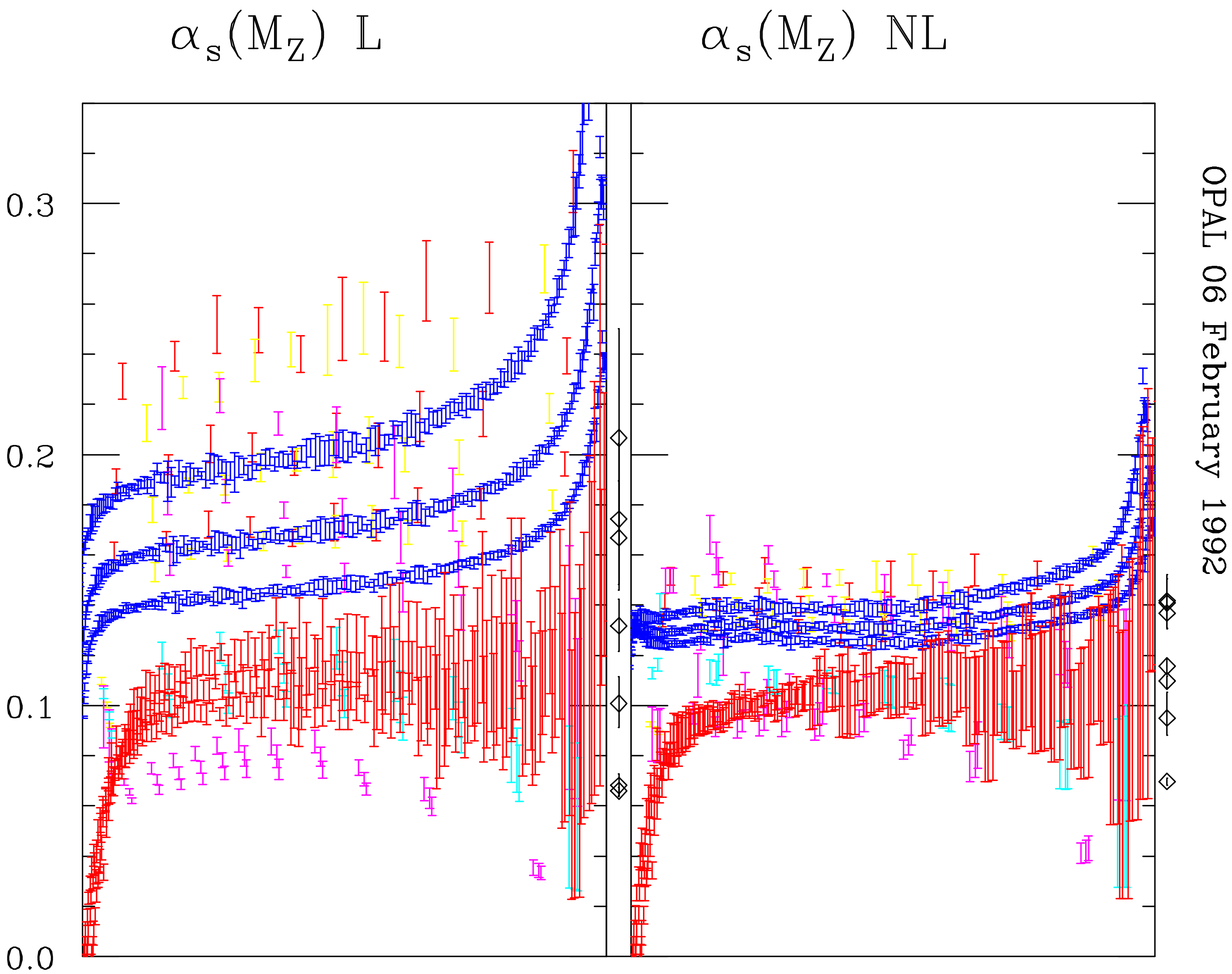
It is clear that inclusive quantities are correctly computed using this procedure. However, the net finite result is the sum of a positive infinite result and a negative infinite one. In practice, this causes difficulties in the convergence of the calculations (opposite spikes in nearby bins)



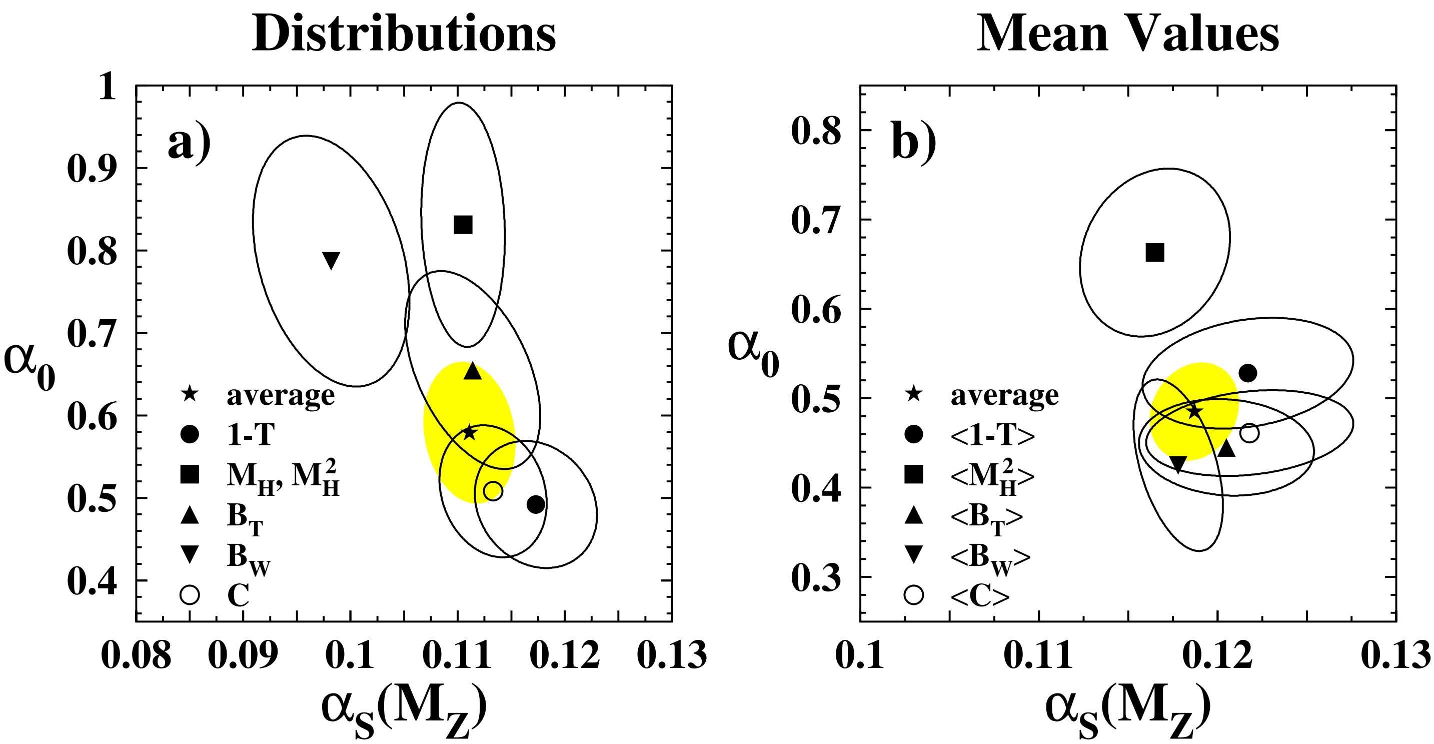
Much work needed to get good convergence properties. The slicing method has (unavoidably) the same problem (but does not show it).

Do we need to go NNLO?

One example: shape variables at LEP



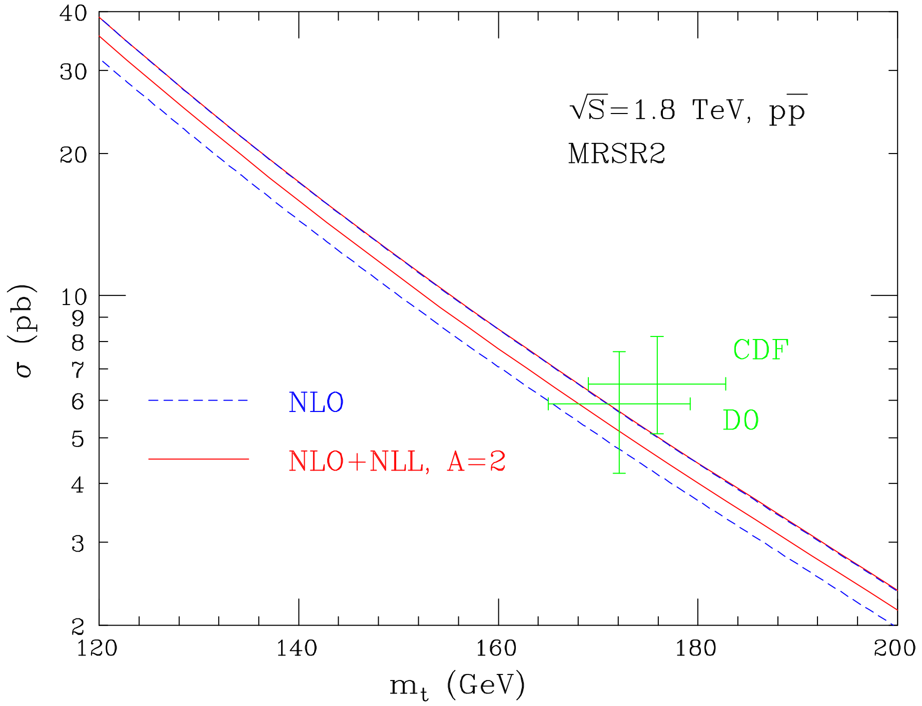
Increased consistency from LO to NLO hints to very important NNLO effects.

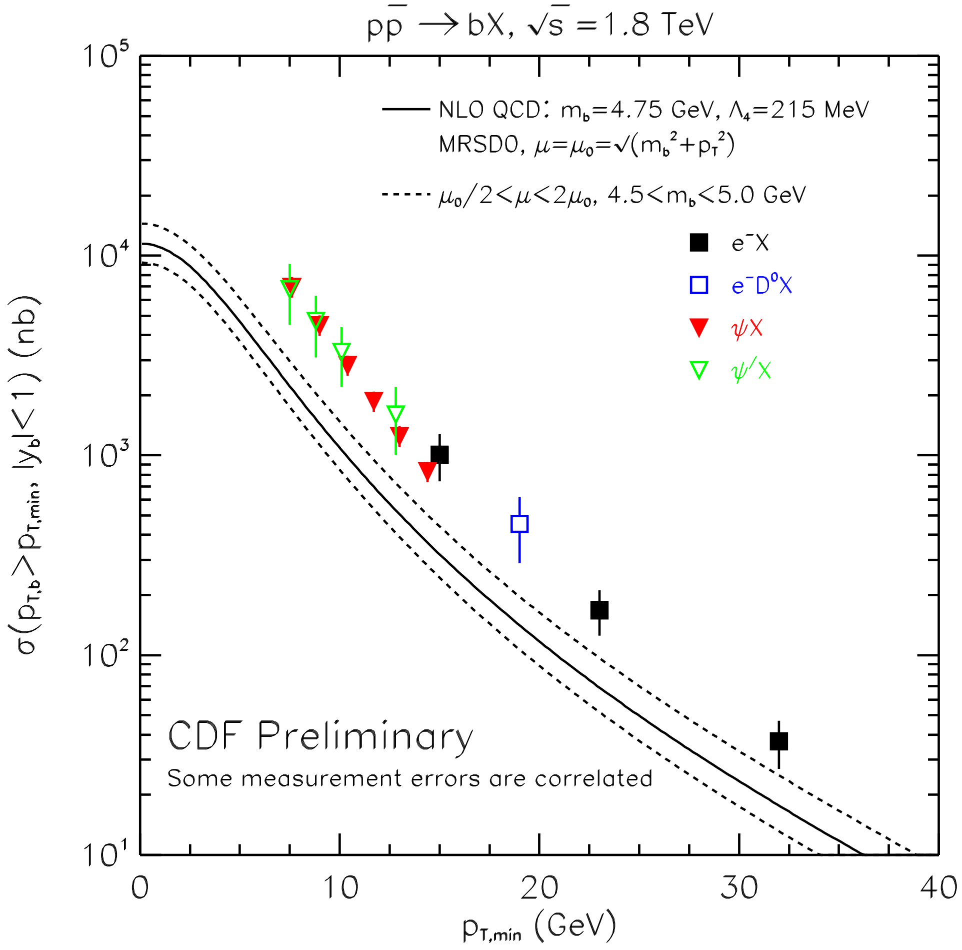


Power effects may still be very important at LEP energy.

However NNLO effects cannot be so small...

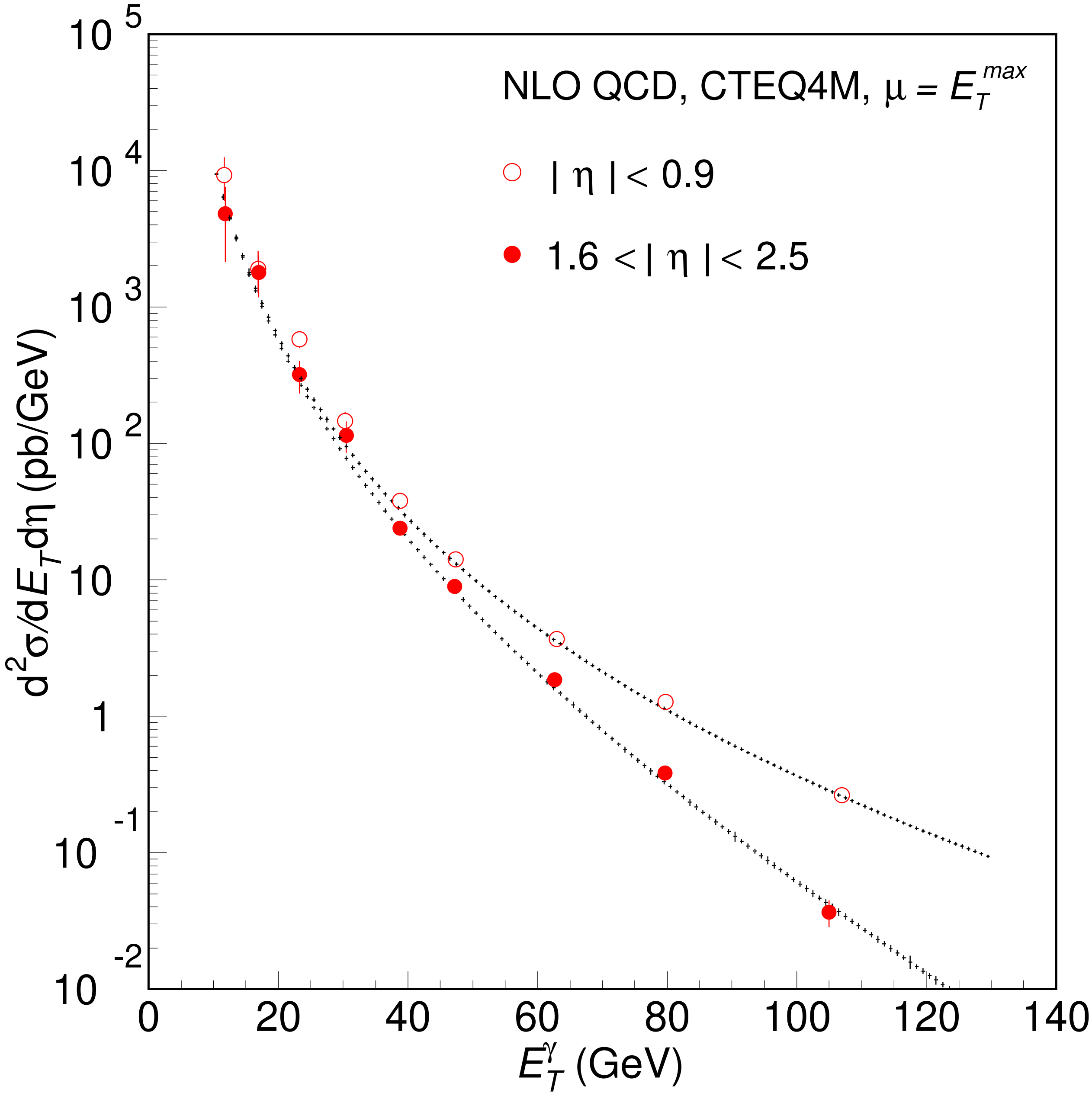
In collider physics, many places with marginal consistency.

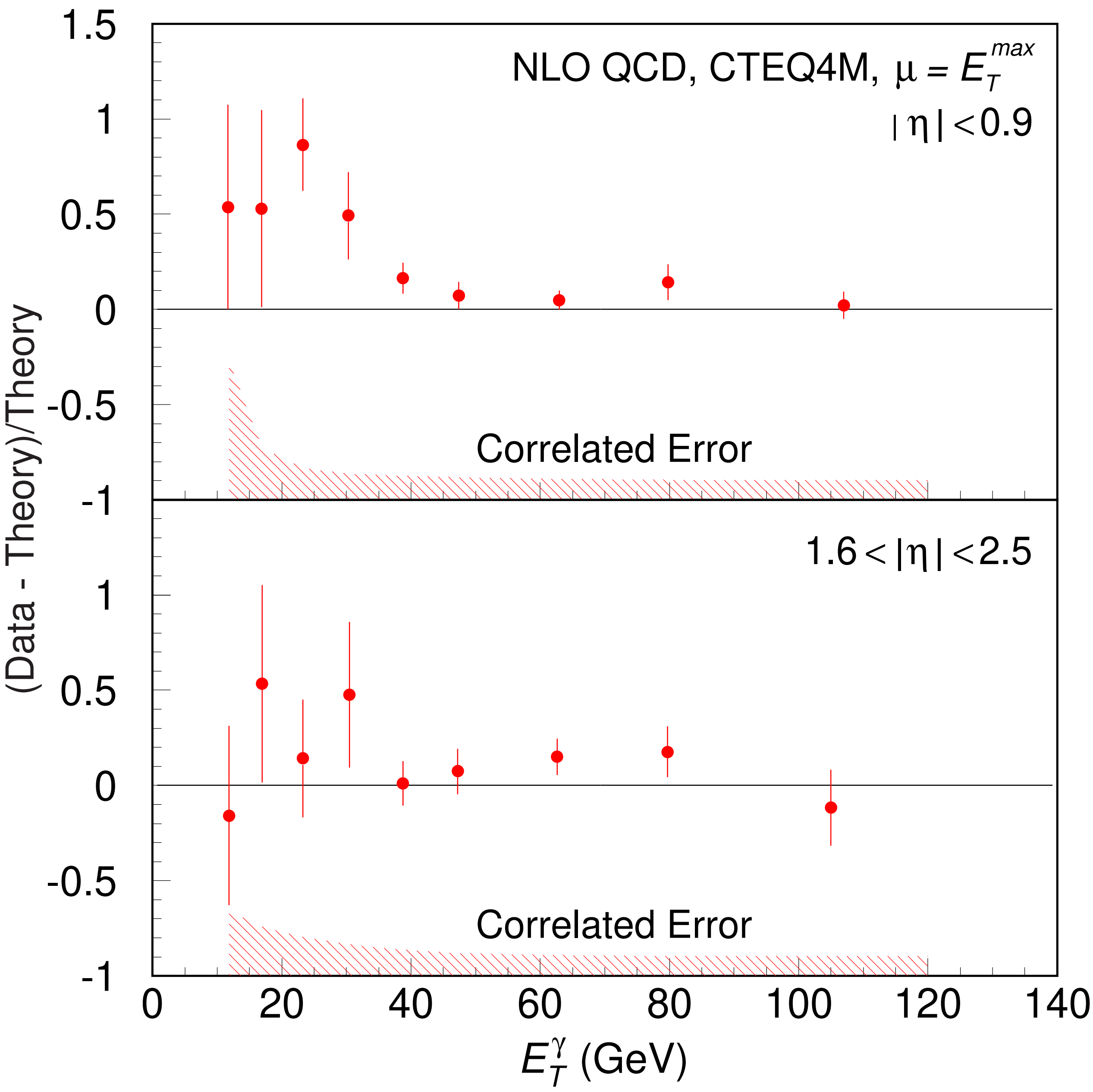




in heavy flavour production, for example, top works, bottom not quite. Radiative corrections are $\approx 100\%$...

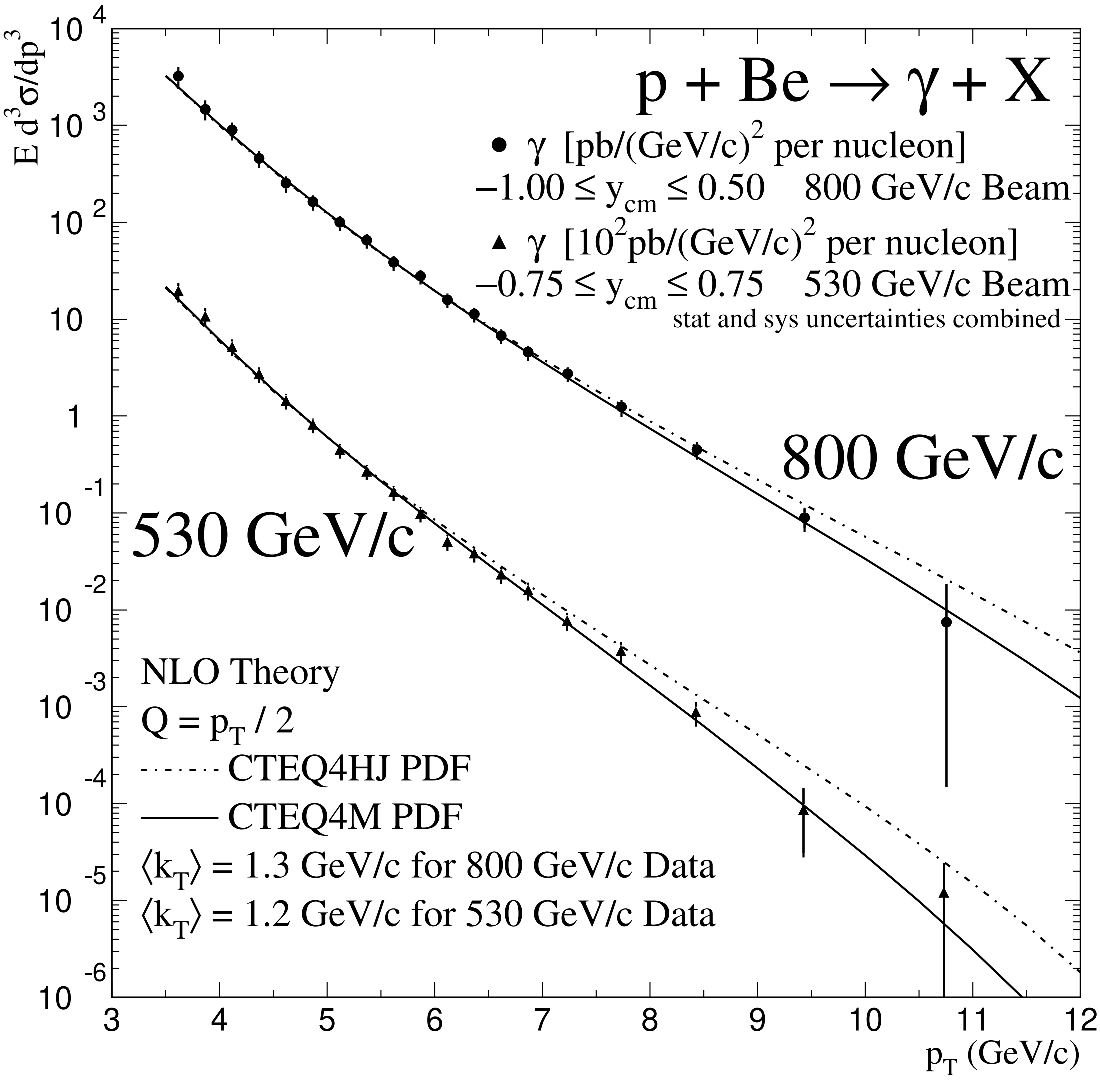
Direct photon production





Inconsistencies at low transverse momentum;

fixed target data requires fixing with an energy dependent transverse momentum kick



This problem affects the extraction of the gluon density from direct photon data, and thus affects QCD tests with very high energy jets at the Tevatron.

Status of NNLO calculations The focus is on jet production in hadronic collision, that is to say, on the parton-parton scattering process (jet production in e^+e^- annihilation is next in complexity). One needs

- the square of the $2{\to}4$ tree amplitude
- the interference of the $2{\to}3$ tree level and one loop amplitude
- the square of the $2{\to}2$ one loop amplitude
- the interference of the $2{\to}2$ two loop and tree level amplitude
- for a consistent phenomenological treatement, the $\mathcal{O}(\alpha_{\rm S}^3)$ Altarelli-Parisi splitting functions are also needed

For the last item, van Neerven and A. Vogt have derived approximate expressions, based upon constraints coming from the large and small x behaviour, and the moments known at the NNLO level. Vermaseren and Moch are working on a formalism for the full, exact computation (for now they have rederived in a simple way the two loop result of Curci, Furmanski and Petronzio).

Techniques for the computation of tree level amplitude for the $2 \to 4$ process have been available for a long time. The problem is the collinear and soft limits of such amplitudes, that must be regularized in order to implement the cancellation and factorization of soft and collinear divergences. In dimensional regularization, these singularities will appear as poles in $1/\epsilon$ up to the fourth power. Therefore, the structure of these singularities must be understood with the required accuracy in ϵ, in order to get a correct result after cancellation. Indeed, all these results are available now

- Double soft limit (Berends and Giele, 1989)
  when two final state particles become soft. Can yield singularities up to $1/\epsilon^4$ after final state integration of the soft particles

- Double collinear and soft-collinear (Campbell and Glover, 1998; Catani and Grazzini, 1999; Del Duca, Frizzo and Maltoni, 2000)
  a subset of 3 final state partons become collinear, or 2 become collinear and one is soft (up to $1/\epsilon^3$ singularities)

The computation of $2 \to 3$ amplitude at one loop level are also known (Bern, Dixon and Kosower, 1993; Kunszt, Signer and Trócsányi, 1994). Again, the collinear and soft limit of this amplitude is needed in order to meet the accuracy required by NNLO calculations

- Collinear limit of one loop amplitudes (Bern, Dixon, Dunbar and Kosower, 1994; Kosower, 1999; Kosower and Uwer, 1999)
- Soft limit of one loop amplitudes (Bern, Del Duca, Schmidt, 1998; Bern, Del Duca, Kilgore and Schmidt 1999; Catani and Grazzini, 2000)
- Some analytic phase space integrals (Gehrman, Glover for $e^+e^- \to \gamma + \mathrm{jet}$ at $\mathcal{O}(\alpha\alpha_\mathrm{s})$; Catani, de Florian, Grazzini for Higgs production at hadron colliders)

The two loop $2\to 2$ contribution has been recently computed (Anastasiou, Glover, Oleari, Tejeda-Yeomans; 2001). Recursion relations among feynman integrals are used to reduce the problem to the computation of a small number of master integrals. The hardest of those (double box, planar and crossed) have been solved only recently (Smirnov, 1999; Tausk, 1999). The structure of the $1/\epsilon$ singularities of these amplitude can be checked against a general factorization formula (Catani, 1998).

All ingredient needed for a full NLO computation of jet cross sections in hadronic collisions are in place. The implementation of the various terms into a useful result still appears as a formidable task (even the one-loop case required much work).
It appears now, however, that (perhaps for LHC times) a result will become available in useful time.

NLO and shower montecarlo

NLO and NNLO calculation do not give the best possible description of the final state.

- Only suitably averaged final state quantities are calculable (only IR safe quantities)
- They give negative contributions in some regions of phase space

From a theoretical viewpoint, we know that the presence of negative values in the cross section could be avoided by resummation of enhanced all order graphs. This would also give a better description of the final state.

Shower Montecarlo programs do precisely that. However, they have been developed essentially at the LO (more precisely LL) level. Merging them with NLO results seems to be a difficult (but not impossible) task

From a practical viewpoint

- Experimentalists prefer implementation of theoretical results as generators of unweighted events, to be fed into the detector simulation machinery. Because of the high cost of detector simulation, weighted events, usually generated in implementation of NLO calculation, are not practical for them.
- Interfacing NLO calculation with some parton shower and hadronization model is also of practical importance.

What was commonly done up to now is to use NLO calculation to fix the normalization of certain inclusive distributions, but use the LL Montecarlo to simulate the full event.
Attempts to overcome the first problem (Dobbs) have been implemented, that use the inclusive character of many experimental measurement to implement real and virtual cancellation. In the case of W pair production, followed by semileptonic decay of the W, the neutrino is not detected, and jets are detected only above a certain energy. Integrating the real and virtual contributions in the unmeasured kinematical variables yields finite results (although not always positive). This approach is however not always applicable, and it does not resolve the second problem.

Shower Montecarlo approach

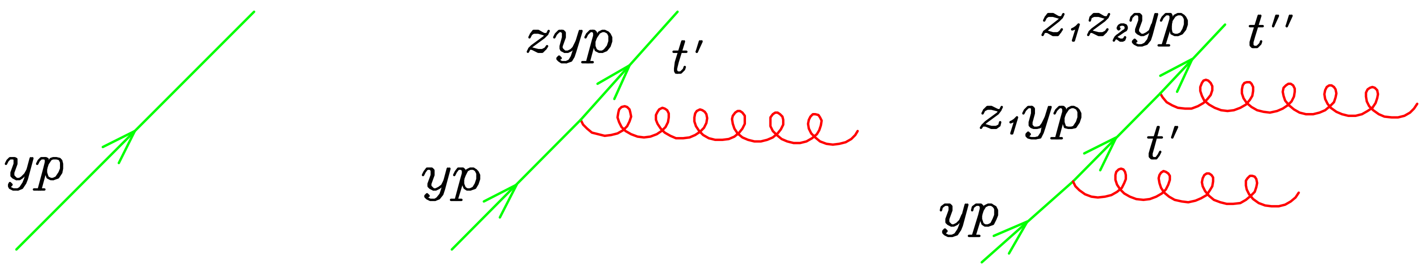
Write the AP equations in integral form

$$f_i(x,t) = f_i(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_S(t')}{2\pi} P_{ij}(z) f_j(y,t') \delta(x-yz)\, dy\, dz \tag{1}$$

and solve it by iteration

$$\begin{aligned}
f_i(x,t) &= f_i(x,t_0) + \\
&\int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_S(t')}{2\pi} P_{ij}(z) f_j(y,t_0) \delta(x-yz)\, dy\, dz + \\
&\int_{t_0}^{t} \frac{dt'}{t'} \frac{\alpha_S(t')}{2\pi} \int_{t_0}^{t'} \frac{dt''}{t''} \frac{\alpha_S(t'')}{2\pi} \times \\
&\qquad P_{ij}(z_2) P_{jk}(z_1) f_k(y,t_0) \delta(x-yz_1z_2)\, dy\, dz_1 dz_2 + \ldots
\end{aligned}$$

Each term of the right hand side correspond to a fixed order contribution in perturbation theory (with running α_S!)



Can easily convince yourselves, using

$$\alpha_S(t) = \frac{\alpha_S(t_0)}{1 + b_0\alpha_S(t_0)\log t/t_0} \tag{2}$$

that we are summing up all terms of the form $(\alpha_S \log t/t_0)^n$ (Leading Log approximation).

The graphs contributing to the AP equations seem to represent uncorrelated emission processes. This is in fact not quite the case: virtual interference effects are present in the splitting functions, and contribute at the point $z = 1$. These contributions spoil the positivity of the splitting functions.

It is easy to reconstruct the effect of the virtual contributions using unitarity.

An intuitive picture of the parton density is given in terms of the wavefunction of the incoming particle. If we probe the particle for a short time $\approx 1/Q$, we may find it in a virtual state, in which it has split into more particles. These states last for a time $1/\Delta E$, and thus, as we increase Q, more and more phase space becomes available for them. The total probability to find the particle in any state must however sum up to one.

Thus, for example, if a splitting process exists that takes particle i into the pair jk, a corresponding correction that reduces the $i \to i$ splitting must also exists.

Focus on qq component:

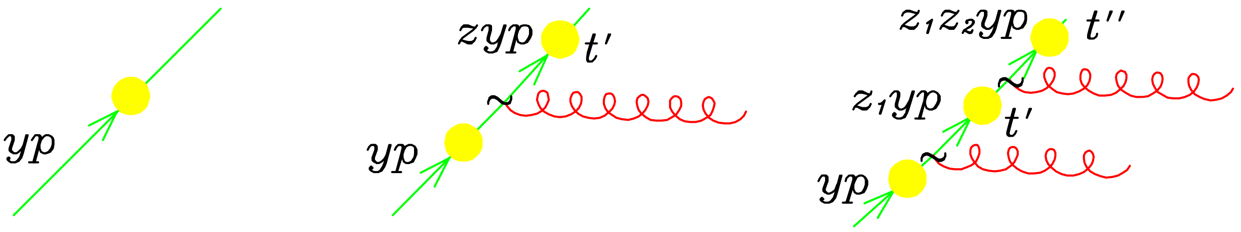
$$P_{qq}(z) = \tilde{P}_{qq}(z) - \delta(1-z)\int \tilde{P}_{qq}(z)\,dz \tag{3}$$

($\tilde{P}$ stands for the real part only) so that

$$\int P_{qq}(z)dz = 0\,, \tag{4}$$

which is consistent with the net charge conservation.

The iterative solution of the AP equation can also be written in terms of the $\tilde{P}$ (real only) splitting functions



$+ \;\ldots$

where the splitting function have been substituted by the real ($\tilde{P}$) ones, and the remaining terms (those proportional to $\delta(1-z)$, and diagonal in parton species) have been accumulated into the yellow blobs, called Sudakov form factors, and given by

$$\begin{aligned}\Delta(t_0,t) &= 1 - \int_{t_0}^{t}\frac{dt'}{t'}\frac{\alpha_S(t')}{2\pi}\hat{P} + \\ &\int_{t_0}^{t}\frac{dt'}{t'}\frac{\alpha_S(t')}{2\pi}\int_{t_0}^{t'}\frac{dt''}{t''}\frac{\alpha_S(t'')}{2\pi}\hat{P}^2 + \ldots \\ &= \exp\left[-\int\frac{dt'}{t'}\frac{\alpha_S(t')}{2\pi}\hat{P}\right]\end{aligned} \tag{5}$$

where $\hat{P} = \int dz \tilde{P}(z)$.

When written in this way, the AP equation can be implemented as a probabilistic branching process, with the Sudakov form factors interpreted as the probability of no emission from the virtuality t_0 to the virtuality t.
We generate a random number r, and solve the equation

$$r = \Delta(t_0, t_1) \tag{6}$$

for the first emission virtuality t_1. We associate to it a splitting process, generating z according to the distribution $P(z)$. Then we iterate. If the generated t exceeds the process scale (Q^2 in the case of DIS) we stop, and we assume that the given parton, with a given momentum enters into the hard scattering process we are considering, which is again simulated by Monte Carlo techniques.

In a full shower Montecarlo program all splitting processes are considered, and final state particles are also followed for collinear timelike branchings using similar techniques. When small virtualities are reached, some hadronization model completes the description of the final state.

The most popular Montecarlo programs:

ISAJET (F. Paige), COJET (R. Odorico), FIELDAJET (R. Field), implement initial and final state branching, as well as an independent framentation model for hadronization

JETSET (T. Sjöstrand) uses a string hadronization scheme to describe hadron production in e^+e^- annihilation; the branching process include optional angular ordering to improve the soft gluon emission description. PYTHIA and LEPTO implement the same hadronization scheme for hadron-hadron and lepton-hadron collisions. ARIADNE (L. Lönnblad) also uses the JETSET hadronization, with an alternative description of the parton shower (the so called dipole picture).

HERWIG (Marchesini and Webber) implements a low-mass cluster hadronization scheme. The perturbative branching includes angular ordering effects, and azimuthal correlations.

The showe Montecarlo is accurate in the region where virtualities (transverse momenta) are strongly ordered (it sums up correctly all strong corrections of the form $(\alpha_{\mathrm{S}} \log Q^2)^k$).
What does it mean to go beyond LO?

- Include also all terms $\alpha_{\mathrm{S}}(\alpha_{\mathrm{S}} \log Q^2)^k$; this is a formidable task, requiring the inclusion of NLO branching processes, multiple branching, and NLO corrections at the interaction vertex
- Include NLO corrections at the interaction vertex, including exact matrix element for the emission of the hardest parton
- Make at least sure that hard emission is correctly described

The second alternative seems to be viable (although not easy)

- Evolution in modern shower Montecarlo does not start from a parton density at a low scale; goes backward, from a parton density at a high scale. Thus, the error on evolution does not affect the rate.
- In this approach, the jet structure would only be accurate at LL, but the emission of the hardest jet would still be accurate at NLO

A satisfactory answer has not yet appeared
Some results are available for the third item.

Some approaches

B.Webber has proposed a method to generate hard multijet configuration which are at least correct at leading order, and incorporate jet fragnentation at NNLO accuracy.

One borrows a clustering algorithm from e^+e^- jet physics (k_T clustering). Multijet cross sections have been computed in next-to-leading logarithmic order in the clustering parameter. For example, the two jet rate in this approximation is given by

$$\begin{aligned} R_2(Q_1, Q) &= |\Delta_q(Q_1, Q)|^2 \, , \\ \Delta_q(Q_1, Q) &= \exp\left(- \int_{Q_1}^{Q} dq \Gamma_q(q, Q) \right) \, , \\ \Gamma_q(q, Q) &= \frac{2C_F}{\pi} \frac{\alpha_S(q)}{q} \left(\log \frac{Q}{q} - \frac{3}{4} \right) \end{aligned}$$

where $Q_1^2 = yQ^2$ and y is the chosen cluster parameter. The two jet fraction is essentially a Sudakov form factor for no emission of resolvable jets with a y parameter above the cut. The generation of jet clusters is performed according to the exact n-parton matrix elements, supplemented with the appropriate Sudakov factor and running couplings. The remaining structure of the event is generated applying the showering algorithm to each cluster, modified in such a way that no overcounting is done in the showering, and the y dependence of the cross section cancels in the appropriate way at the end. This is done by evolving the final state cascade from a scale Q, and vetoing the branchings with scale $q > Q_1$.

Seymour has proposed a method for correcting the parton shower when hard emissions are present. Essentially, events are reweighted with the exact matrix element when there is a hard emission. To avoid inconsistencies, reweighting is performed at each branching, if that branching is the hardest so far. This method has been applied to top decay and vector boson production (Corcella and Seymour).

Collins has proposed a method for correcting the shower at the NLO level. This is done by adding the NLO contribution, after subtraction of the approximate emission probability that is used in the shower model, to avoid overcounting. The method has been applied so far only to the qg contribution to the W production cross section (no soft divergences...).

We may expect progress in this field in the near future...

Soft Gluon Effects

When approaching the end of phase space, enhanced contributions arise to all order in perturbation theory.

In Drell-Yan ($\tau = Q^2/s$)

$$\sigma(\tau, Q^2) = \int_0^1 dx\, dx_1\, dx_2 F(x_1)F(x_2)\delta(xx_1x_2 - \tau)\Delta(x, Q^2)$$

At the Born level $\Delta(x, Q^2) = \delta(1-x)$. The limit $\tau \to 1$ is better studied in N space for large N

$$f_N = \int_0^1 \frac{dz}{z}\, z^N\, f(z)$$

$$\sigma_N(Q^2) = F_N^2(Q^2)\, \Delta_N(Q^2)\,.$$

The resummed coefficient function for the Drell–Yan process in the DIS scheme is (Sterman, Catani and Trentadue)

$$\ln \Delta_N(Q^2) = -\int_0^1 dx\, \frac{x^N - 1}{1-x} \left[2\int_{(1-x)^2Q^2}^{(1-x)Q^2} \frac{dq^2}{q^2} A(\alpha_S(q^2)) \right.$$

$$\left. + B(\alpha_S((1-x)Q^2)) \right] + \mathcal{O}(\alpha_S(\alpha_S \ln N)^k) \;\;,$$

$$A(\alpha_S) = \frac{\alpha_S}{\pi} A^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 A^{(2)}\,, \qquad B(\alpha_S) = \frac{\alpha_S}{\pi} B^{(1)}\,.$$

In double log approximation

$$\ln \Delta_N(Q^2) = \frac{A^{(1)}}{\pi} \alpha_S \ln^2 N + \mathcal{O}((\alpha_S \ln N)^{k+1} \ln N)$$

Results known for quite some time; lots of problems in the interpretation

- Korchemski and Sterman have tried to use these formulae for describing the renormalon structure of the DY cross section (when the $\alpha_S(q^2)$ integration goes over the Landau pole), exposing a $1/Q$ power correction. In fact, there is no reason to believe the resummation formulae in this region. Beneke and Brown have explicitly proved in one example (large n_f limit) that the $1/Q$ term cancels when all (not only collinear enhanced) soft singularities are included.

- The same formulae, transformed to x space, gives rise to unjustifiable factorial growth, not related to renormalons, giving fractional power corrections in $1/Q$ (Catani, Mangano, Trentadue, P.N.). This is best seen from the double log approximation

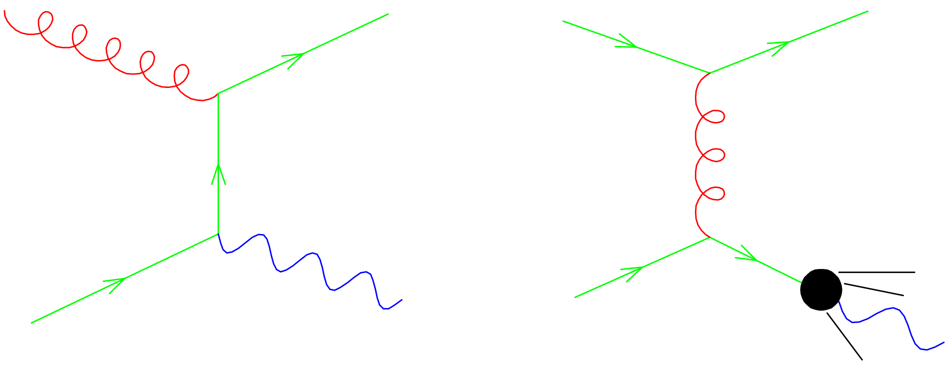
  $$\Delta_N(Q^2) = \exp(a \ln^2 N) \rightarrow \Delta(x, Q^2) = \exp(a \ln^2(1-x))$$

  (the first expression has finite value, the moments of the second expression are infinite)

There is not yet consensus over the answer to these problems, although it certainly seems that one cannot use Sudakov resummation to decide about non perturbative behaviour.

Some (less discussed) problem arise when we remember that also the large x behaviour of the parton densities come into play when comparing different subprocesses.

Large x_T direct photon production (traditionally one of the most important tool to measure the gluon structure function at large x values) is one example.



One usually says that fragmentation is suppressed at large x_T (the scaled photon transverse energy) in this process.

One takes moments in x_T, and looks at large N,

$$\begin{aligned}
\sigma_{\gamma,\,N}(E_T) &\equiv \int_0^1 dx_T^2\;(x_T^2)^{N-1}\;E_T^3\,\frac{d\sigma_\gamma(x_T,E_T)}{dE_T} \\
&= \sum_{a,b} f_{a/H_1}^{(N+1)}\;f_{b/H_2}^{(N+1)}\Big\{\hat{\sigma}_{ab\to\gamma,\,N} \\
&+ \sum_c \hat{\sigma}_{ab\to c,\,N}\;d_{c/\gamma,\,2N+3}\Big\}\;.
\end{aligned}$$

What prevails for large N depends only upon the large N behaviour of the parton densities and of the fragmentation function. We have

$$\frac{d}{dt}\begin{vmatrix} S_N \\ G_N \end{vmatrix} = \begin{vmatrix} A_N^{qq} & A_N^{qg} \\ A_N^{gq} & A_N^{gg} \end{vmatrix} \begin{vmatrix} S_N \\ G_N \end{vmatrix},$$

$$t = \frac{1}{2\pi}\int d\log Q^2 \alpha(Q^2) = \frac{1}{2\pi b_0}\log\frac{\alpha(Q_0^2)}{\alpha(Q^2)},$$

S_N and G_N are moments of the singlet and the gluon parton densities, and A_N^{ab} are the moments of the Altarelli–Parisi splitting functions that for large N, have the behaviour

$$A_N^{qq} \approx -2C_{\mathrm{f}}\log N\,, \quad A_N^{gg} \approx -2C_{\mathrm{A}}\log N\,,$$
$$A_N^{gq} \approx \frac{C_{\mathrm{f}}}{N}\,, \quad A_N^{qg} \approx \frac{n_{\mathrm{f}}}{N}\,.$$

Assuming that at the initial condition G_N is not larger than S_N, it is easy to get asymptotically

$$\frac{G_N(t)}{S_N(t)} = \frac{A_N^{gq}}{A_N^{qq} - A_N^{gg}} \approx \frac{1}{N\log N}\frac{C_{\mathrm{f}}}{2(C_{\mathrm{A}} - C_{\mathrm{f}})} = \frac{4}{10}\frac{1}{N\log N}\,.$$

An analogous argument gives for the asymtotics of the fragmentation function

$$D_N^{\gamma q}(t) = \frac{e_q^2}{2C_{\mathrm{f}}}\frac{1}{N\log N}\frac{\alpha_{\mathrm{em}}}{\alpha(t)}$$

Let us consider the case of proton-antiproton collisions. The large N behaviour of the $q\bar{q}$, qg and fragmentation subprocesses is given by

$$\begin{aligned}
\sigma_N^{q\bar{q}} &\approx C_N \alpha \alpha_{\rm em} S_N^2 \\
\sigma_N^{qg} &\approx C_N \alpha \alpha_{\rm em} S_N G_N \\
\sigma_N^{q \to \gamma} &\approx C_N \alpha^2 S_N^2 D_N^{\gamma q}
\end{aligned} \tag{7}$$

where C_N is the large N behaviour of the partonic cross sections. So: the qg and fragmentation components have the same asymptotic behaviour. In the case of proton-antiproton collisions, the $q\bar{q}$ mechanism dominates. However, in the interesting case of proton-proton collisions it does not, and one is left with the qg and the fragmentation process that compete at the same level.

One may ask if Sudakov effects may decide which mechanism dominates. However this is not enough. Double logarithms do not appear in the diagonal splitting functions, but they may appear in the off diagonal ones

$$\begin{aligned} A_N^{gq} &\rightarrow A_N^{qg}\left(1+\frac{\alpha}{2\pi}(C_\mathrm{A}-C_\mathrm{f})\log^2 N\right) \\ A_N^{\gamma q} &\rightarrow A_N^{\gamma g}\left(1+\frac{\alpha}{2\pi}C_\mathrm{f}\log^2 N\right) \end{aligned}$$

which propagate into the answer, and compete with Sudakov factors. Other logarithmic enhancements arise in the NL cross section for the process $qq' \rightarrow qq\prime\gamma$. At the end one finds

$$\frac{qg}{\mathrm{fragmentation}} = \frac{\Delta_g}{\Delta_q^2}\left(1-\frac{\alpha(t)}{2\pi}(C_\mathrm{A}-2C_\mathrm{f})\log^2 N\right) \ .$$

with

$$\Delta_{g/q} \approx \exp\left(\frac{\alpha_\mathrm{S}}{2\pi}2C_{A/F}\log^2 N\right) \ .$$

Thus the Sudakov effects seem to be reduced by other kinds of double logarithms.

In general, the Sudakov formalism we use now does not seem to be capable of answering questions that depend upon the large N behaviour of the parton densities!

Conclusions

- Considerable progress in NNLO calculations, the answer is not too far
- A beginning in NLL Shower Montecarlo
- Many difficult, deep theoretical question waiting to be answered in resummation techniques