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CHARACTERIZING AND MEASURING MULTIPARTITE ENTANGLEMENT

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A method is proposed to characterize and quantify multipartite entanglement in terms of the probability density function of bipartite entanglement over all possible balanced bipartitions of an ensemble of qubits. The method is tested on a class of random pure states.

Keywords: Entanglement; randomness; cluster states.

The quantification of multipartite entanglement is an open and very challenging problem. An exhaustive definition of *bipartite* entanglement exists and hinges upon the von Neumann entropy and the entanglement of formation,[1,2] but the problem of defining *multipartite* entanglement is more difficult[3] and no unique definition exists: different definitions tend indeed to focus on different aspects of the problem, capturing different features of entanglement,[4–6] that do not necessarily agree with each other. Moreover, as the size of the system increases, the number of measures (i.e. real numbers) needed to quantify multipartite entanglement grows exponentially.

This work is motivated by the idea that a good definition of multipartite entanglement should stem from some statistical information about the system.[7] We shall therefore look at the distribution of the purity of a subsystem over all bipartitions of the total system. As a measure of multipartite entanglement we will take a whole *function*: the probability density of bipartite entanglement between any two parts of the total system. According to our definition multipartite entanglement is large when bipartite entanglement (i) is large *and* (ii) does not depend on the bipartition, namely, when (i) and (ii) the probability density of bipartite entanglement is a narrow function centered at a large value. This definition will be tested on two class of states that are known to be characterized by a large entanglement. We emphasize that the idea that complicated phenomena cannot be "summarized" in a single (or a few) number(s) was already proposed in the context of complex systems[8] and has been also considered in relation to quantum entanglement.[9]

We shall focus on a collection of n qubits and consider a partition in two subsystems A and B, made up of n_A and n_B qubits ($n_A + n_B = n$), respectively. For definiteness we assume $n_A \leq n_B$. The total Hilbert space is the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the dimensions are $\dim \mathcal{H} = N = 2^n$, $\dim \mathcal{H}_A = N_A = 2^{n_A}$ and $\dim \mathcal{H}_B = N_B = 2^{n_B}$, respectively ($N_A N_B = N$).

We shall consider pure states. Their expression adapted to the bipartition reads

$$|\psi\rangle = \sum_{k=0}^{N-1} z_k |k\rangle = \sum_{j_A=0}^{N_A-1} \sum_{l_B=0}^{N_B-1} z_{j_A l_B} |j_A\rangle \otimes |l_B\rangle, \tag{1}$$

where $|k\rangle = |j_A\rangle \otimes |l_B\rangle$, with a bijection between k and (j_A, l_B). Think of the binary expressions of an integer k in terms of the binary expression of (j_A, l_B).

As a measure of *bipartite* entanglement between A and B we consider the participation number

$$N_{AB} = \pi_{AB}^{-1}, \quad \pi_{AB}(|\psi\rangle) = \mathrm{tr}_A \rho_A^2, \quad \rho_A = \mathrm{tr}_B \rho, \tag{2}$$

where $\rho = |\psi\rangle\langle\psi|$ and tr_A (tr_B) is the partial trace over the subsystem A (B). N_{AB} measures the effective rank of the matrix ρ_A, namely the effective Schmidt number.[10] We note that

$$1 \leq N_{AB} = N_{BA} \leq \min(N_A, N_B), \tag{3}$$

with the maximum (minimum) value attained for a completely mixed (pure) state ρ_A. Therefore, a larger value of N_{AB} corresponds to a more entangled bipartition (A, B), the maximum value being attainable for a *balanced* bipartition, i.e. when $n_A = [n/2]$ (and $n_B = [(n+1)/2]$), where $[x]$ is the integer part of the real x, that is the largest integer not exceeding x. The maximum possible entanglement is $N_{AB} = N_A = 2^{n_A}$. The quantity $n_{AB} = \log_2 N_{AB}$ represents the effective number of entangled qubits, given the bipartition (namely, the number of bipartite entanglement "links" that are "severed" when the system is bipartitioned).

Clearly, the quantity N_{AB} will depend on the bipartition, as in general entanglement will be distributed in a different way among all possible bipartitions. As explained in the introduction, we are motivated by the idea that the distribution $p(N_{AB})$ of N_{AB} yields information about *multipartite* entanglement.

Let us therefore study the typical form of our measure of multipartite entanglement $p(N_{AB})$ for a very large class of pure states, sampled according to a given symmetric distribution on the projective Hilbert space $\{\psi \in \mathcal{H}, \|\psi\| = 1\}$ (e.g. the unitarily invariant Haar measure). By plugging (1) into (2), one gets

$$\pi_{AB} = \sum_{j,j'=0}^{N_A-1} \sum_{l,l'=0}^{N_B-1} z_{jl} \bar{z}_{j'l} z_{j'l'} \bar{z}_{jl'}, \tag{4}$$

and it can be shown[7] that, in the thermodynamical limit (that is practically attained for $n > 5$), independently of the distribution of the coefficients, the mean and the standard deviation of (4) over all possible balanced bipartitions read

$$\mu_{AB} = \frac{N_A + N_B - 1}{N} = \sqrt{\frac{\alpha}{N}}, \quad \sigma^2_{AB} = \frac{2}{N^2}, \quad (N \text{ large}) \tag{5}$$

respectively, where $\alpha = 8/2$ ($\alpha = 9/2$) for even (odd) n. Moreover, the probability density of N_{AB} in Eq. (2) reads

$$p(N_{AB}) = \frac{1}{N^2_{AB}(2\pi\sigma^2_{AB})^{1/2}} \exp\left(-\frac{(N^{-1}_{AB} - \mu_{AB})^2}{2\sigma^2_{AB}}\right). \tag{6}$$

It is interesting to compare the features of these generic random states with those of other states studied in the literature. Table 1 displays the average value of N_{AB} for GHZ states,[11] W states,[12,13] cluster states[14] and the generic states (1), for $n = 5 \div 12$. While the entanglement of the GHZ and W states is essentially independent of n, the situation is drastically different for cluster and random states. In both cases, the average entanglement increases with n; for $n > 8$, the average entanglement is higher for random states. However, the mean $\langle N_{AB} \rangle$ yields poor information on multipartite entanglement. For this reason, it is useful to analyze the distribution of bipartite entanglement over all possible balanced bipartitions.

The results for the cluster and random states are shown in Fig. 1, for n ranging between 5 and 12. Notice that the distribution function of the random state is always peaked around $\langle N_{AB} \rangle \simeq \mu^{-1}_{AB}$ given by (5). Notice also that the cluster state can reach higher values of N_{AB} (the maximum possible value being $2^{[n/2]}$), however, the fraction of bipartitions yielding this result becomes smaller for higher n. This is immediately understood if one realizes that the cluster states are designed for optimized applications and therefore perform better in terms of *specific* bipartitions. On the other hand, according to the measure we propose, the random states are characterized by a large value of multipartite entanglement, that is roughly *independent* of the bipartition.

Table 1. Mean bipartite entanglement $\langle N_{AB} \rangle$ for different states and different number of qubits n.

n	GHZ	W	Cluster	Random
5	2	1.923	3.6	2.909
6	2	2	5.4	4.267
7	2	1.96	6.171	5.565
8	2	2	8.743	8.258
9	2	1.976	10.349	10.894
10	2	2	14.206	16.254
11	2	1.984	17.176	21.558
12	2	2	23.156	32.252

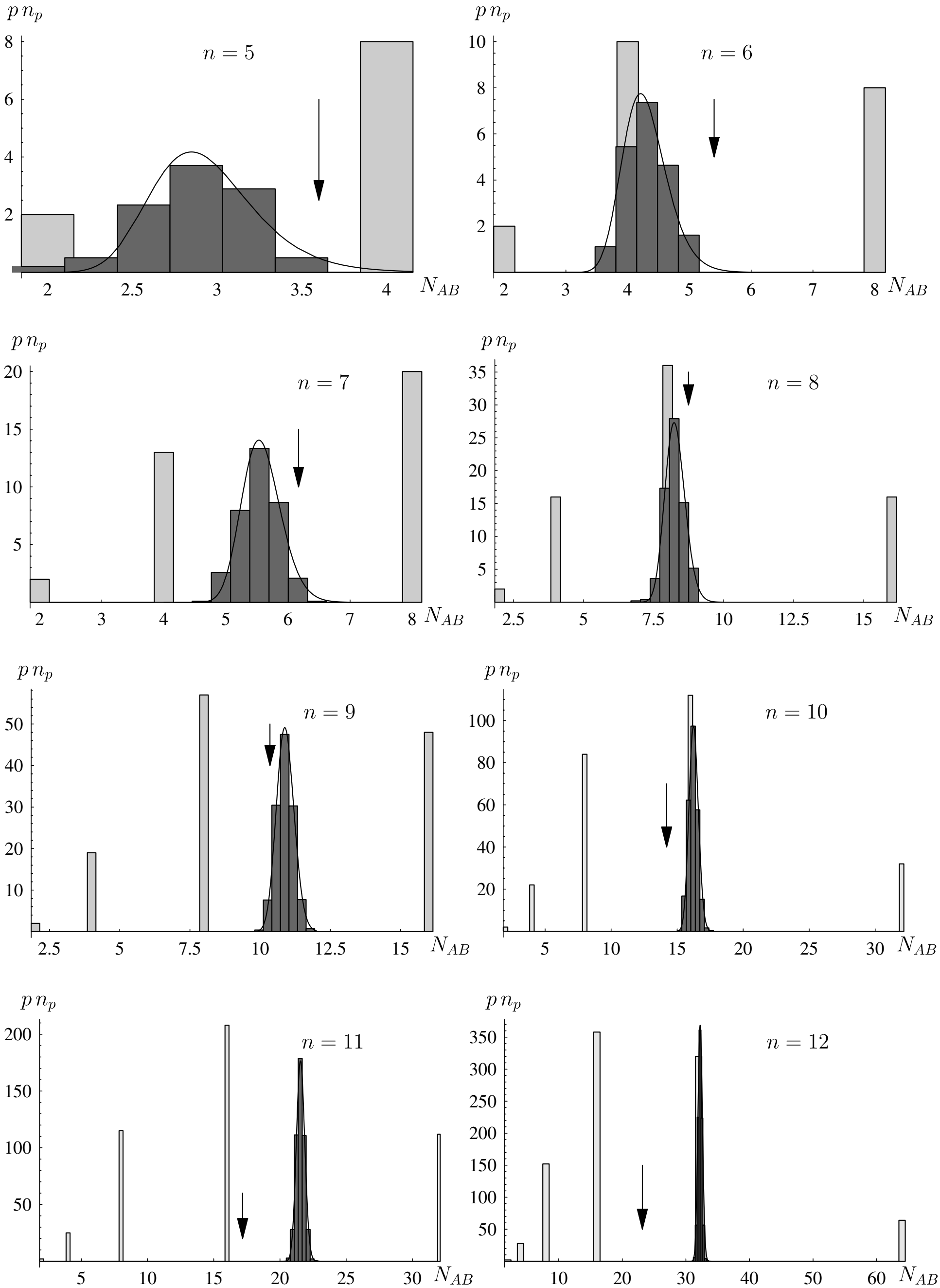
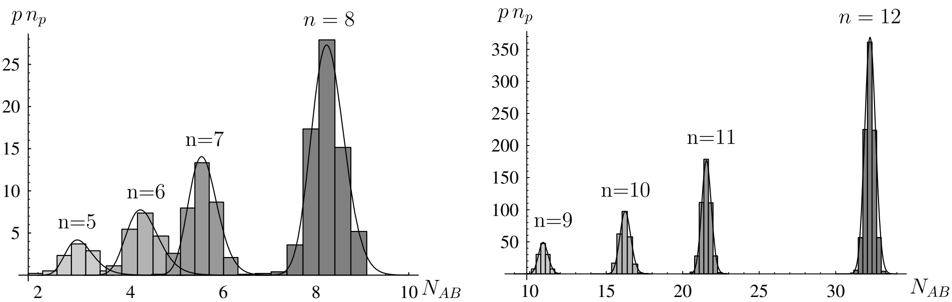


Fig. 1. Number of balanced bipartitions versus N_{AB}; $n_p = n!/n_A!n_B!$ is the number of bipartitions. The light-gray bars represent cluster states, the dark-gray ones random states; the solid line is the distribution (5) and (6); the black arrows indicate the average $\langle N_{AB}\rangle_{\rm cluster}$. For even n ($n = 12$ in particular) the distribution of the random state partially hides a peak of the corresponding cluster state distribution, centered at $N_{AB} = 2^{n_A-1} = 2^{[n/2]-1}$.

In Fig. 2, we compare the number of balanced bipartitions versus N_{AB} for the random states and increasing n. The related probability density functions (6) are displayed in Fig. 3. Notice that as the number of spins increases from $n = 5$ to $n = 12$, the mean increases and the distribution becomes relatively narrower. As we emphasized, these are both signatures of a very high degree of multipartite entanglement, whose features become (as n increases) practically independent of the bipartition. In Fig. 3, it is interesting to observe the difference between the distributions for odd and even n.

In Fig. 4(a), we plot the value of $\langle N_{AB}\rangle$ for the cluster and random states (see Table 1). We notice that, for $n = 9$, $\langle N_{AB}\rangle_{\text{random}}$ becomes larger than $\langle N_{AB}\rangle_{\text{cluster}}$. Figure 4(b) displays the behavior of the standard deviation of N_{AB},

$$\sigma \simeq \sigma_{AB}/\mu_{AB}^2. \tag{7}$$



(a) (b)

Fig. 2. Number of balanced bipartitions versus N_{AB}; the histograms are numerically obtained for the typical states; the solid line represents their distribution. The value of n (total number of spins) is always indicated and ranges (a) from 5 to 8; (b) from 9 to 12; $n_p = n!/n_A!n_B!$ is the number of bipartitions.

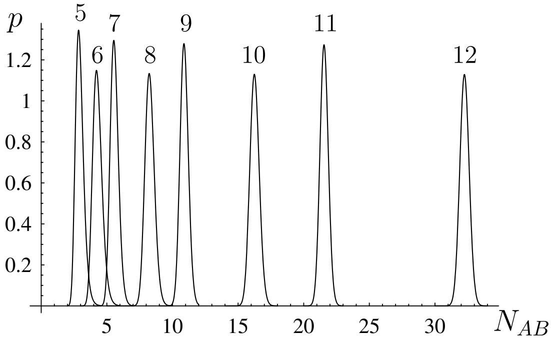


Fig. 3. Probability densities functions (6) versus N_{AB}. Each curve is labeled with the corresponding value of n (number of qubits). The standard deviation σ quickly becomes independent of n [see Fig. 4(b)] and depends only on the parity of the latter.

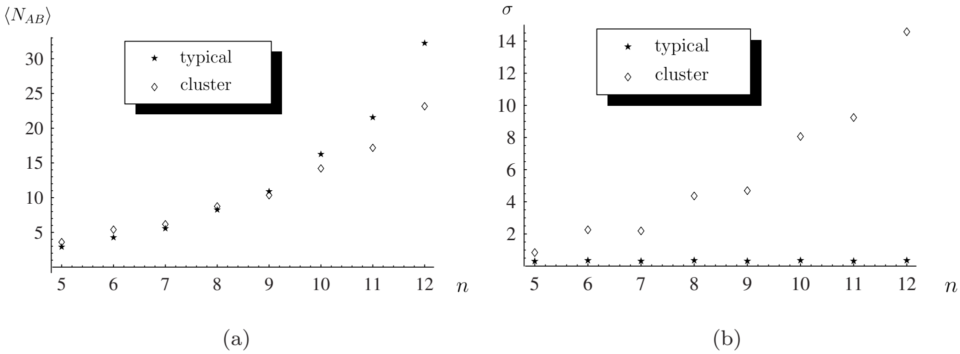


Fig. 4. Comparison between typical and cluster states: (a) expectation value $\langle N_{AB}\rangle$ and (b) standard deviation σ of the distributions in Fig. 3 versus n (number of qubits).

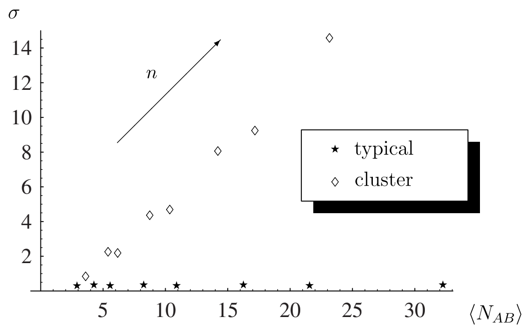


Fig. 5. Standard deviation σ versus expectation value $\langle N_{AB}\rangle$ of the distributions in Fig. 3.

For the cluster states this quantity tends to diverge when the size of the system increases. By contrast, from Eq. (5), $\sigma = \sqrt{2}/\alpha$ is constant for the typical states. This means that the ratio $\sigma/\langle N_{AB}\rangle$ tends to 0. Finally, Fig. 5 displays a parametric plot of σ versus $\langle N_{AB}\rangle$. Clearly, for the random states σ is independent of $\langle N_{AB}\rangle$.

We emphasize that our analysis should by no means be taken as an argument against the performance of the cluster states. As we stressed before, cluster states are tailored for specific purposes in quantum information processing, and in that respect are very well suited. We compared the generic states to the cluster states specifically because the latter are also known to be characterized by a large entanglement.

An efficient way to generate states endowed with random features is by means of a chaotic dynamics,[20–23] or at the onset of a quantum phase transition.[24–27] In particular, the random states describe quite well states with support on chaotic regions of phase space, before dynamical localization has taken place. These features make these states rather appealing, from a practical point of view, in that they are easily generated. The introduction of a probability density function as a measure of multipartite entanglement paves the way to further investigations of the intimate relation between entanglement and randomness and their behavior across a phase transition.

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