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QP–PQ
Quantum Probability and White Noise Analysis
Volume XIX

Quantum Information and Computing

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World Scientific

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INVARIANT SUBSPACES AND CONTROL OF DECOHERENCE

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We discuss three different control strategies, all aimed at countering the effects of decoherence. The first strategy hinges upon the quantum Zeno effect, the second makes use of frequent unitary interruptions ("bang-bang" pulses), and the third of a strong, continuous coupling. Decoherence is suppressed if the frequency N of the measurements/pulses is large enough or if the coupling K is sufficiently strong. However, if N or K are large, but not extremely large, all these control procedures accelerate decoherence.

1. Introduction

The deterioration of the coherence features of quantum systems, due to their interaction with the environment, is known as decoherence [1] and represents the most serious obstacle against the preservation of quantum superpositions and entanglement over long periods of time. The possibility of controlling (and eventually halting) decoherence is a key problem with important applications, e.g. in quantum computation [2]. We focus here on three schemes that have been recently proposed in order to counter the effects of decoherence. The first is based on the quantum Zeno effect (QZE) [3,4,5]), the second on "bang-bang" (BB) pulses and their generalization, quantum dynamical decoupling [6] and the third on a strong, continuous coupling (when this can be viewed as a measurement of some sort [7]). These apparently different methods are in fact related to each other [8] and a sistematic study of their analogies and differencies helps understanding under which circumstances and physical conditions all these controls may *accelerate*, rather than hinder decoherence.

In this paper we will outline the main results of a comparison among these control strategies (the complete proofs can be found in [9]). We stress that the notion of "bang-bang" control is well known in engineering and in connection with spin-echo techniques [10], so that this control can be considered "classical," its revival in quantum-information-related problems being very recent. Moreover, the idea that a strong continuous interaction with an external field or "apparatus" may be viewed as a measurement on the system and can slow its dynamics is not new [7]. (In fact, this turns out to be one of the most efficient control procedures.) For the above-mentioned reasons, the similarities among the three control methods are not surprising and their comparison interesting.

Our main objective is to endeavor to understand in which sense one can *control* decoherence [11] and to outline the key role played by the form factors of the interaction. The method we propose is general and can be applied to diverse situations of practical interest, such as atoms and ions in cavities, organic molecules, quantum dots and Josephson junctions [12].

2. Generalities and notation

Let the total system consist of a target system S and a reservoir B and its Hilbert space be expressed as the tensor product $\mathcal{H}_{\rm tot} = \mathcal{H}_S \otimes \mathcal{H}_B$. The total Hamiltonian

$$H_{\rm tot} = H_0 + H_{SB} = H_S + H_B + H_{SB} \tag{1}$$

is the sum of the system Hamiltonian H_S, the reservoir Hamiltonian H_B and their interaction H_{SB}, which is responsible for decoherence; the operators H_S and H_B act on $\mathcal{H}_S$ and $\mathcal{H}_B$, respectively. H_0 is the free total Hamiltonian.

The dynamics of the total system is conveniently reexpressed in terms of the Liouvillian

$$\mathcal{L}_{\rm tot}\rho \equiv -i[H_{\rm tot}, \rho] = -i\left(H_{\rm tot}\rho - \rho H_{\rm tot}\right) , \tag{2}$$

where ρ is the density matrix. If the Hamiltonian is given by (1), the Liouvillian is accordingly decomposed into

$$\mathcal{L}_{\rm tot} = \mathcal{L}_0 + \mathcal{L}_{SB} = \mathcal{L}_S + \mathcal{L}_B + \mathcal{L}_{SB}, \tag{3}$$

where the meaning of the symbols is obvious.

We assume that the interaction Hamiltonian H_{SB} in (1) can be written as [13]

$$H_{SB} = \sum_m \left(X_m \otimes A_m^\dagger + X_m^\dagger \otimes A_m\right) , \tag{4}$$

where the X_m are the eigenoperators of the system Liouvillian, satisfying

$$\mathcal{L}_S X_m = i\omega_m X_m \qquad (\omega_m \neq \omega_n, \quad \text{for} \quad m \neq n) \tag{5}$$

and A_m are the destruction operators of the bath

$$A_m = A(g_m) = \int d^3k\, g_m^*(\boldsymbol{k})\, a(\boldsymbol{k}) , \tag{6}$$

expressed in terms of bosonic operators $a(\boldsymbol{k})$, with form factors $g_m(\boldsymbol{k})$.

The bare spectral density functions (form factors) read

$$\kappa_m(\omega) = \int d^3k\, |g_m(\boldsymbol{k})|^2 \delta(\omega_k - \omega) , \tag{7}$$

with $\kappa_m(\omega) = 0$, for $\omega < 0$, and the thermal spectral density functions $[N(\omega) = 1/(e^{\beta\omega} - 1)$, where β is the inverse temperature$]$

$$\kappa_m^\beta(\omega) = \kappa_m(\omega)\,(N(\omega)+1) + \kappa_m(-\omega)N(-\omega) = \frac{1}{1-e^{-\beta\omega}}\left[\kappa_m(\omega) - \kappa_m(-\omega)\right]$$

extend along the whole real axis, due to the counter-rotating terms, and satisfy the KMS symmetry [14]

$$\kappa_m^\beta(-\omega) = \frac{N(\omega)}{N(\omega)+1}\,\kappa_m^\beta(\omega) = \exp(-\beta\omega)\,\kappa_m^\beta(\omega). \tag{8}$$

We focus on two particular (Ohmic) cases: an exponential form factor

$$\kappa_m^{(E)}(\omega) = g^2\omega \exp(-\omega/\Lambda)\theta(\omega) \tag{9}$$

and a polynomial form factor

$$\kappa_m^{(P)}(\omega) = g^2 \frac{\omega}{[1+(\omega/\Lambda)^2]^n}\theta(\omega), \tag{10}$$

where g is a coupling constant, Λ a cutoff and θ the unit step function. In order to properly compare these two cases, we require that the bandwidth be the same

$$W = \frac{\int_{-\infty}^{\infty} d\omega\, |\omega|\kappa_m^{(E)}(\omega)}{\int_{-\infty}^{\infty} d\omega\, \kappa_m^{(E)}(\omega)} = \frac{\int_{-\infty}^{\infty} d\omega\, |\omega|\kappa_m^{(P)}(\omega)}{\int_{-\infty}^{\infty} d\omega\, \kappa_m^{(P)}(\omega)}. \tag{11}$$

We focus on a proper subspace $\mathcal{H}_{\rm comp} \subset \mathcal{H}_S$, in which quantum computation is to be performed. For this reason we look in detail at the case

$$\mathcal{H}_S = \mathcal{H}_{\rm comp} \oplus \mathcal{H}_{\rm orth} \tag{12}$$

and consider for simplicity a single qubit, $\mathcal{H}_{\rm comp} = \mathbb{C}^2$.

The initial state of the total system $\rho(0)$ is taken to be the tensor product of the system and reservoir initial states

$$\rho(0) = \sigma(0) \otimes \rho_B \ , \tag{13}$$

where the reservoir equilibrium state has an inverse temperature β

$$\rho_B = \frac{1}{Z} \exp(-\beta H_B), \qquad (\mathcal{L}_B \rho_B = 0) \tag{14}$$

and $Z = \mathrm{tr}_B e^{-\beta H_B}$ is the normalization constant.

The system state $\sigma(t)$ at time t is given by the partial trace

$$\sigma(t) \equiv \mathrm{tr}_B \rho(t). \tag{15}$$

There is decoherence when $\sigma(t)$ is not unitarily equivalent to $\sigma(0)$ for a given class of initial states. In the Markov approximation the state of the system (15) satisfies the master equation

$$\dot{\sigma}(t) = (\mathcal{L}_S + \mathcal{L})\, \sigma(t) \ ,$$

where, up to a renormalization of the free Liouvillian $\mathcal{L}_S$ by Lamb and Stark shift terms, $\mathcal{L}$ engenders the dissipation due to the interaction with the bath,

$$\begin{aligned}\mathcal{L}\sigma = \gamma_0 &\left(X_0 \sigma X_0 - \frac{1}{2} \{X_0 X_0, \sigma\} \right) \\ &+ \sum_{m \geq 1} \gamma_m \left(X_m \sigma X_m^\dagger - \frac{1}{2} \{X_m^\dagger X_m, \sigma\} \right) \\ &+ \sum_{m \geq 1} \gamma_{-m} \left(X_m^\dagger \sigma X_m - \frac{1}{2} \{X_m X_m^\dagger, \sigma\} \right) \ ,\end{aligned} \tag{16}$$

where

$$\gamma_m = 2\pi \kappa_m^\beta (\omega_m) \tag{17}$$

are the decay rates.

A particular case of the above is the qubit Hamiltonian

$$H_{SB} = \sigma_z \otimes \left[A(g_0) + A^\dagger(g_0)\right] + \sigma_x \otimes \left[A(g_1) + A^\dagger(g_1)\right], \qquad H_0 = \frac{\Omega}{2} \sigma_z \ . \tag{18}$$

This is of the form (4), when one identifies

$$X_0 = \sigma_z, \quad X_{\pm 1} = \sigma_{\mp} = \frac{\sigma_x \mp i\sigma_y}{2}, \qquad \omega_{\pm 1} = \pm\Omega, \quad \omega_0 = 0,$$

hence

$$\begin{aligned}\mathcal{L}\rho = \gamma_0 \left(\sigma_z \rho \sigma_z - \rho\right) + \gamma_{+1} &\left(\sigma_- \rho \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho\} \right) \\ + \gamma_{-1} &\left(\sigma_+ \rho \sigma_- - \frac{1}{2} \{\sigma_- \sigma_+, \rho\} \right) \ ,\end{aligned}$$

with

$$\gamma_0 = 2\pi \kappa_0^\beta (0), \qquad \gamma_{\pm 1} = 2\pi \kappa_1^\beta (\pm\Omega) \ . \tag{19}$$

3. Control procedures

3.1. *Quantum Zeno control*

In general, the purpose of the control is to suppress decoherence, as expressed by the "unitarity defect" of the evolution (15). We first look at the Zeno control, by adapting the argument of Ref. [15]. The control is obtained by performing frequent measurements of the system:

$$\rho \longrightarrow \hat{P}\rho \equiv \sum_n P_n \rho P_n, \tag{20}$$

where $\hat{P}$ is a projection superoperator and $\{P_n\}$ a complete ($\sum_n P_n = \mathbf{1}_S$) set of orthogonal projection operators acting on $\mathcal{H}_S$. We restrict our analysis to a measuring apparatus that does not "select" the different outcomes (nonselective measurement) [16]. The measurement is designed so that

$$\hat{P} H_{SB} = \sum_n P_n H_{SB} P_n = 0 \quad \Leftrightarrow \quad \hat{P} \mathcal{L}_{SB} \hat{P} = 0. \tag{21}$$

We will see that a similar requirement is necessary for the other control procedures, to be analyzed in the next subsections. The Zeno control consists in performing repeated nonselective measurements at times $t = k\tau$ ($k = 0, 1, 2, \ldots$) (we include an initial "state preparation" at $t = 0$). Between successive measurements, the system evolves via H_{tot}. The density matrix after $N+1$ measurements, with an initial state $\rho(0)$, in the limit $\tau \to 0$ while keeping $t = N\tau$ constant, reads

$$\begin{aligned}\rho(t) = \rho(N\tau) &= \left[\hat{P} e^{\mathcal{L}_{\mathrm{tot}} \tau} \hat{P}\right]^N \rho(0) \\ &= \hat{P} \left[1 + \hat{P} \mathcal{L}_{\mathrm{tot}} \hat{P} \tau + \mathrm{O}\left(\tau^2\right)\right]^{\frac{t}{\tau}} \rho(0) \stackrel{\tau \to 0}{\longrightarrow} \hat{P} e^{\hat{P} \mathcal{L}_{\mathrm{tot}} \hat{P} t} \rho(0) \ ,\end{aligned}$$

where the controlled Liouvillian is

$$\mathcal{L}_{\mathrm{tot}}' = \hat{P} \mathcal{L}_{\mathrm{tot}} \hat{P} = \hat{P} \mathcal{L}_S \hat{P} + \mathcal{L}_B \hat{P} \ . \tag{22}$$

Hence, as a result of infinitely frequent measurements, the system-reservoir coupling is eliminated and, thus, decoherence is halted. Also, transitions among different sectors of the system Hilbert space, defined by the measurement superoperator $\hat{P}$, become forbidden, yielding a superselection rule and the formation of invariant "Zeno" subspaces [15]. The "decoherence-free" subspace [17] is one of these Zeno subspaces.

We assume for simplicity that $\hat{P}$ commutes with the system Liouvillian

$$\hat{P}\mathcal{L}_S = \mathcal{L}_S\hat{P}, \tag{23}$$

so that

$$\mathcal{L}'_{\text{tot}} = (\mathcal{L}_S + \mathcal{L}_B)\hat{P} . \tag{24}$$

The crucial issue is to understand what happens when the interval $\tau = t/N$ between measurements is *finite*. The evolution of the reduced state operator is governed by

$$\dot{\sigma}(t) = [\mathcal{L}_S + \mathcal{L}_Z(\tau)]\, \sigma(t) ,$$

where the dissipative part is found to have the explicit form [analogous to Eq. (16)]

$$\begin{aligned}\mathcal{L}_Z(\tau)\sigma = \gamma_0^Z(\tau)\hat{P}\left(X_0\hat{P}\sigma X_0 - \frac{1}{2}\left\{X_0X_0, \hat{P}\sigma\right\}\right)\\ + \sum_{m\geq 1}\gamma_m^Z(\tau)\hat{P}\left(X_m\hat{P}\sigma X_m^\dagger - \frac{1}{2}\left\{X_m^\dagger X_m, \hat{P}\sigma\right\}\right)\\ + \sum_{m\geq 1}\gamma_{-m}^Z(\tau)\hat{P}\left(X_m^\dagger\hat{P}\sigma X_m - \frac{1}{2}\left\{X_m X_m^\dagger, \hat{P}\sigma\right\}\right) ,\end{aligned} \tag{25}$$

with the controlled decay rates

$$\gamma_m^Z(\tau) = \tau\int_{-\infty}^{\infty} d\omega\, \kappa_m^\beta(\omega)\, \text{sinc}^2\left(\frac{\omega-\omega_m}{2}\tau\right) , \tag{26}$$

where $\text{sinc}(x) = \sin(x)/x$. Let us focus on the exponential (9) and polynomial form factors (10). We work in the high-temperature case, which is rather critical from an experimental point of view, because of temperature-induced transitions in two-level systems, and set $\Omega = 0.01W$, $\beta = 50W^{-1}$, so that temperature $= \beta^{-1} = 2\Omega$. Observe that

$$\gamma^Z(\tau) \sim \frac{\tau}{\tau_Z^2}, \quad (\tau \to 0) \tag{27}$$

$$\tau_Z^{-2} = \int_{-\infty}^{\infty} d\omega\, \kappa^\beta(\omega) = \int_0^\infty d\omega\, \kappa(\omega)\coth\left(\frac{\beta\omega}{2}\right) ,$$

τ_Z being the thermal Zeno time. (We dropped the suffix m for simplicity.) Notice also that

$$\gamma^Z(\tau) \to \gamma, \qquad \tau \to \infty , \tag{28}$$

where

$$\gamma = 2\pi\kappa^\beta(\Omega) \tag{29}$$

is the natural decay rate (17). The ratio $\gamma^Z(\tau)/\gamma$ is the key quantity: decoherence is suppressed (controlled) if $\gamma^Z(\tau) < \gamma$, and it is enhanced otherwise. The latter phenomenon is known in the literature as *inverse* Zeno effect [18,19,20]. The key issue is to understand *how small* τ should be in order to get suppression (control) of decoherence (QZE), rather than its enhancement. This ratio $\gamma^Z(\tau)/\gamma$ is shown in Figs. 1 and 2 as a function of τ [in units W–the bandwidth defined in Eq. (11)]. The transition between the two regimes takes place at $\tau = \tau^*$, where τ^* is defined by the equation [20]

$$\gamma^Z(\tau^*) = \gamma. \tag{30}$$

It is useful to spend a few words on the *physical* meaning of the expressions $\tau \to 0$, $\beta \to \infty$ in the above (and following) formulas. Times and temperatures are to be compared with the bandwidth W (or frequency cutoff Λ). Times (temperatures) are "small" when $\tau \ll W^{-1}$ ($\beta^{-1} \ll W$). For example, when one considers short-time expansions in a Zeno context, the relevant timescale is τ^* [20,21]: the expansion (27) is valid for $\tau \lesssim W^{-1}$ (and not $\tau \lesssim \tau_Z$, as it is sometimes erroneously assumed).

3.2. *Control via quantum dynamical decoupling and "bang-bang" pulses*

We now turn our attention to the so-called quantum dynamical decoupling [6], and in particular to a kicked control ("bang-bang" pulses). In this case, one applies after each time interval τ an *instantaneous* unitary operators U_k and gets the following convenient explicit expression of the effective Hamiltonian [8]

$$H'_{\text{tot}} = \hat{P}H_{\text{tot}} = \sum_n P_n H_{\text{tot}} P_n, \tag{31}$$

where the projections P_n arise from the spectral decomposition

$$U_k = \sum_n e^{-i\lambda_n}P_n, \qquad (\lambda_n \neq \lambda_m \bmod 2\pi, \quad \text{for } n \neq m) . \tag{32}$$

By assuming again, as in (21) and (23), that $[\hat{P}, \mathcal{L}_S] = 0$ and that $\hat{P}\mathcal{L}_{SB}\hat{P} = 0$ we get the controlled evolution

$$\rho(t) = \left[e^{\mathcal{L}_k} e^{\mathcal{L}_{\rm tot}\tau}\right]^{\frac{t}{\tau}} \hat{P}\rho(0) \to e^{\mathcal{L}'_{\rm tot}t}\hat{P}\rho(0), \qquad \tau \to 0\ , \tag{33}$$

where $\mathcal{L}_k$ is the Liouvillian corresponding to the evolution (32) and

$$\mathcal{L}'_{\rm tot} = \hat{P}\mathcal{L}_{\rm tot}\hat{P} = (\mathcal{L}_S + \mathcal{L}_B)\hat{P}, \tag{34}$$

exactly as in (24). As in the case discussed in the previous subsection, one observes the formation of invariant Zeno subspaces: transitions among different subspaces vanish in the $\tau \to 0$ limit, yielding a superselection rule. In this case, the subspaces are defined by (31)-(32) and are nothing but the ergodic sectors of U_k. Note also that the controlled Liouvillians for bang bang pulses, (34), and for the Zeno control, (24), coincide when the set of orthogonal projections (20) is chosen equal to the set (32) of eigenprojections of U_k, namely

$$\mathcal{L}_k\hat{P} = 0, \qquad (\hat{P}\mathbf{1}) = \mathbf{1}. \tag{35}$$

Therefore, the two controls are equivalent in the ideal (limiting) case [8]. However, throughout this article, by dynamical decoupling we will refer to a situation where the evolution is coherent (unitary), while by Zeno control to a situation where the evolution involves incoherent (nonunitary) processes, such as quantum measurements.

As in the Zeno control, let us look at the τ-finite situation. Let us consider the two level system (18) with $g_0 = 0$ (spin-flip decoherence). We include an additional third level—that performs the control—and add to (18) the Hamiltonian (acting on $\mathcal{H}_S \oplus \mathrm{span}\{|M\rangle\}$)

$$H_M = -\frac{\Omega}{2}|M\rangle\langle M|, \tag{36}$$

so that $|M\rangle$ is degenerate with $|\downarrow\rangle$. The control consists of a sequence of 2π pulses [22] between $|\downarrow\rangle$ and $|M\rangle$, given by

$$U_k = \exp\left[-i\pi\left(|\downarrow\rangle\langle M| + |M\rangle\langle\downarrow|\right)\right] = P_\uparrow - P_{-1}\ , \tag{37}$$

where

$$P_\uparrow = |\uparrow\rangle\langle\uparrow|, \qquad P_{-1} = P_\downarrow + P_M = |\downarrow\rangle\langle\downarrow| + |M\rangle\langle M|, \tag{38}$$

are the eigenprojections of U_k (belonging respectively to $e^{-i\lambda_\uparrow} = 1$ and $e^{-i\lambda_{-1}} = -1$), that define two Zeno subspaces.

One gets for the decay rate out of state $|\uparrow\rangle$

$$\begin{aligned}\gamma^k(\tau) &= \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}\right)^2}\left[\kappa^\beta\left(\Omega + \frac{2\pi}{\tau}(j+1/2)\right) + \kappa^\beta\left(\Omega - \frac{2\pi}{\tau}(j+1/2)\right)\right] \\ &\overset{\tau\to 0}{\sim} \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}\right)^2}\,\kappa\left(\frac{\pi}{\tau}(2j+1)\right),\end{aligned} \tag{39}$$

where in the expansion we assumed that β is not too small (as compared to τ). The key issue, once again, is to understand *how small* τ should be in order to get suppression of decoherence (control), rather than its enhancement.

Notice that

$$\gamma^k(\tau) \to \frac{4}{\pi}\kappa^\beta(\Omega)\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}\right)^2} = \gamma, \qquad \tau \to \infty\ .$$

The ratio $\gamma^k(\tau)/\gamma$ is shown in Figs. 1 and 2 as a function of τ. Once again, the transition between the two regimes takes place at $\tau = \tau^*$, where τ^* is defined by the equation

$$\gamma^k(\tau^*) = \gamma. \tag{40}$$

Observe that the mechanism of decoherence suppression (39) is not fully determined by $\mathcal{L}_{\rm tot}$ and $\hat{P}$, in contrast to the Zeno case, and depends also on the details of the Liouvillian $\mathcal{L}_k$.

3.3. *Control via a strong continuous coupling*

The formulation in the preceding subsections hinges upon instantaneous processes, that can be unitary or nonunitary. However, the basic features of the QZE can be obtained by making use of a continuous coupling, when the external system takes a sort of steady "gaze" at the system of interest. The mathematical formulation of this idea is contained in a theorem [15] on the (large-K) dynamical evolution governed by a *generic* Liouvillian of the type

$$\mathcal{L}_K = \mathcal{L}_{\rm tot} + K\mathcal{L}_c. \tag{41}$$

$\mathcal{L}_c$ can be viewed as an "additional" interaction Hamiltonian performing the "measurement" and K is a coupling constant. The evolution reads

$$\rho(t) = e^{(K\mathcal{L}_c + \mathcal{L}_{\rm tot})t}\hat{P}\rho(0) \to e^{\mathcal{L}'_{\rm tot}t}\hat{P}\rho(0), \qquad K \to \infty\ , \tag{42}$$

[see (33)] where the notation is obvious and

$$\mathcal{L}_c\hat{P} = 0, \qquad (\hat{P}\mathbf{1}) = \mathbf{1} \tag{43}$$

[see (35)].

The above statements can be proved by making use of the adiabatic theorem [23]. Once again, like in the two previous subsections, one observes the formation of invariant Zeno subspaces, that are in this case the eigenspaces of the interaction (43). The links between the quantum Zeno effect and the notion of "continuous coupling" to an external apparatus or environment has often been proposed in the literature of the last 25 years [7]. The novelty here lies in the gradual formation of the Zeno subspaces as K becomes increasingly large. In such a case, they are nothing but the adiabatic subspaces.

In general, as in the BB control but in contrast to the Zeno case, the mechanism of decoherence suppression is not fully determined by H_S and depends on the details of the Hamiltonians H_S *and* H_c. Once again, this can be clarified by looking at a specific example: consider the two level system (18) with $g_0 = 0$ (spin flip decoherence). We add to (18) the Hamiltonian (acting on $\mathcal{H}_S \oplus \mathrm{span}\{|M\rangle\}$)

$$\begin{aligned} H_M &= -\frac{\Omega}{2}|M\rangle\langle M| + KH_c, \\ H_c &= |\downarrow\rangle\langle M| + |M\rangle\langle\downarrow| = P_+ - P_- \,, \end{aligned} \tag{44}$$

where

$$P_\pm = \frac{(|\downarrow\rangle \pm |M\rangle)(\langle\downarrow| \pm \langle M|)}{2} \equiv |\pm\rangle\langle\pm|. \tag{45}$$

The third state $|M\rangle$ is now "continuously" coupled to state $|\downarrow\rangle$, $K \in \mathbb{R}$ being the strength of the coupling. As K is increased, state $|M\rangle$ performs a better "continuous observation" of $|\downarrow\rangle$, yielding the Zeno subspaces [5]. In terms of its eigenprojections, H_c reads

$$H_c = \eta_\uparrow P_\uparrow + \eta_- P_- + \eta_+ P_+, \tag{46}$$

with $P_\uparrow = |\uparrow\rangle\langle\uparrow|$ and $\eta_\uparrow = 0, \eta_\pm = \pm 1$. In the Zeno limit ($K \to \infty$) the subspaces $\mathcal{H}_\uparrow$, $\mathcal{H}_+$ and $\mathcal{H}_-$ decouple due to wildly oscillating phases $O(K)$. We get

$$\hat{P}H_{SB} = P_\uparrow H_{SB} P_\uparrow + P_- H_{SB} P_- + P_+ H_{SB} P_+ = 0. \tag{47}$$

Therefore in the limit $K \to \infty$, $\gamma_{\pm 1} = 0$ and decoherence is halted.

By diagonalizing the new system Hamiltonian one obtains for the decay rate out of state $|\uparrow\rangle$

$$\begin{aligned} \gamma^c(K) &= \frac{\gamma_+(K) + \gamma_-(K)}{2} = \pi\left(\kappa_1^\beta(\Omega - K) + \kappa_1^\beta(\Omega + K)\right) \\ &\sim \pi\kappa(K)\left(1 + e^{-\beta K}\right) \sim \pi\kappa(K), \end{aligned} \tag{48}$$

for $K \to \infty$. On the other hand,

$$\gamma^c(K) \to \gamma, \qquad K \to 0. \tag{49}$$

Notice that the role of K in this subsection and the role of $1/\tau$ in the previous ones are equivalent. This yields a natural comparison [8] between different timescales (τ for measurements and kicks, $1/K$ for continuous coupling).

The ratio $\gamma^c(K)/\gamma$ is shown in Figs. 1 and 2 as a function of $2\pi/K$. The transition between these two regimes takes now place at $K = K^*$ where K^* is defined by the equation

$$\gamma^c(K^*) = \gamma. \tag{50}$$

3.4. *Comparison among the three control strategies*

There is a clear difference between *bona fide* projective measurements and the other two cases, BB kicks and continuous coupling. In the former case τ^* depends on the global features of the form factor (i.e., its integral). By contrast, in the other two cases τ^* "pick" some particular ("on-shell") value(s). This important difference is due to the different features of the evolution (non-unitary in the first case, unitary in the latter cases). The different features discussed above yield very different outputs, clearly apparent in Fig. 2, that can be important in practical applications: decoherence can be more easily halted by applying BB and/or continuous coupling strategies. These two methods yield values of τ^* (or K^*) that are easier to attain. However, this advantage has a price, because BB and continuous coupling yield a larger enhancement of decoherence for $\tau > \tau^*$, $K < K^*$. The two dynamical methods perform better only when $\tau \lesssim \tau^*$, $K \gtrsim K^*$. This is apparent in Fig. 1. We notice that a strict comparison between continuous coupling and the other two methods is difficult, as it would involve an analysis of numerical factors of order one in the definition of the relevant conversion factors between the frequency of interruptions τ and the coupling K (this factor has been sensibly–but arbitrarily–set equal to 2π in Figs. 1-2).

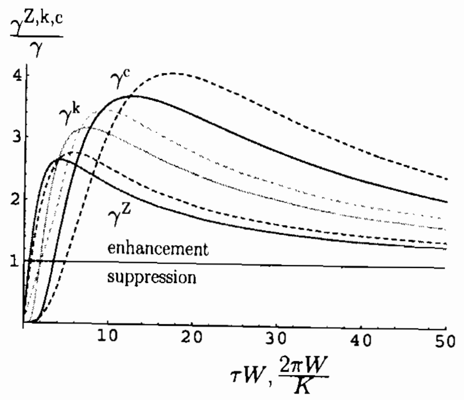

Figure 1. Comparison among the three control methods. The full and dashed lines refer to the exponential and polynomial form factors, respectively. BB kicks and continuous coupling are more effective than *bona fide* measurements for combatting decoherence, as the regime of "suppression" is reached for larger values of τ and K^{-1}.

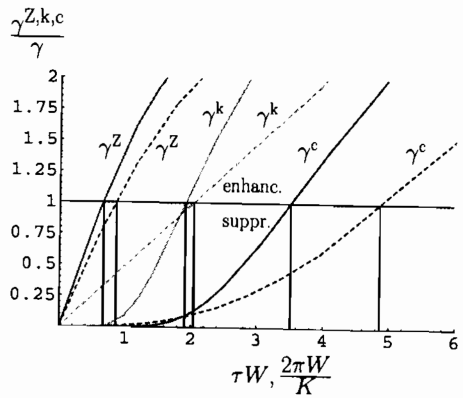

Figure 2. Comparison among the three control methods: small times/strong coupling regions. τ^* and K^* are indicated.

4. The Zeno subspaces

The three different procedures described in the previous section yield, by different physical mechanisms, the formation of invariant Zeno subspaces. This is shown in Fig. 3. If one of these invariant subspaces is the "computational" subspace $\mathcal{H}_{\rm comp}$ introduced in Eq. (12), the possibility arises of inhibiting decoherence in this subspace.

Of course, in principle (in the $\tau, K^{-1} \to 0$ limit), decoherence can be

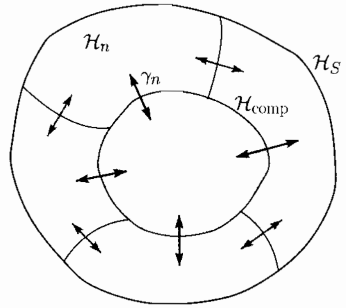

Figure 3. The Zeno subspaces are formed when the frequency τ^{-1} of measurements or BB pulses or the strength K of the continuous coupling tend to ∞. The shaded region represents the "computational" subspace $\mathcal{H}_{\rm comp} \subset \mathcal{H}_S$ defined in Eq. (12). The transition rates γ_n depend on τ or K.

completely *halted*; however, the main objective of our study was to understand *how* the limit is attained and analyze the deviations from the ideal situation. This was done by studying the transition rates γ_n between different subspaces and in particular their τ and K dependence (see Fig. 3). In general, this dependence can be complicated, leading to *enhancement* of decoherence in some cases and *suppression* in other cases. For this reason, the key issue is to understand the *physical* meaning of the expressions $\tau, K^{-1} \to 0$. This point is often sloppily analyzed in the literature.

5. Summary and concluding remarks

We compared three control methods for combatting decoherence. The first is based on repeated quantum measurements (projection operators) and involves a description in terms of nonunitary processes. The second and third methods are both dynamical, as they can be described in terms of unitary evolutions. In all cases, decoherence can be halted by very rapidly/strongly driving or very frequently measuring the system state. However, if the frequency is not high enough or the coupling not strong enough, the controls may accelerate the decoherence process and deteriorate the performance of the quantum state manipulation. The acceleration of decoherence is analogous to the inverse Zeno effect, namely the acceleration of the decay of an unstable state due to frequent measurements [18,19,20].

It is convenient to summarize the main results obtained in this article in the particular case of a two-level system (qubit) with energy difference Ω. If the frequency τ^{-1} of measurements or BB kicks, or the strength K of the coupling tend to ∞, the two-dimensional (Zeno) subspace defining the qubit becomes isolated and decoherence is completely suppressed. However, if τ^{-1} and K are large, but not extremely large, the transition (decay) rates between the qubit subspace and the remaining sector of the Hilbert space display a complicated dependence on τ^{-1} and K, and decoherence can be suppressed or enhanced, depending on the situation.

At low temperatures $\beta^{-1} \ll \Omega \ll W$, where W is the bandwidth of the form factor of the interaction, the decay rates read

$$\begin{cases} \gamma^{Z}(\tau) \sim \frac{\tau}{\tau_Z^2}, & \tau \to 0\,, \\ \gamma^{\mathrm{k}}(\tau) \sim \frac{8}{\pi}\kappa\left(\frac{\pi}{\tau}\right), & \tau \to 0\,, \\ \gamma^{\mathrm{c}}(K) \sim \pi\kappa(K), & K \to \infty\,, \end{cases} \tag{51}$$

where Z, k and c denote (Zeno) measurements, (BB) kicks and continuous coupling, respectively, κ is the form factor and $1/\tau_Z^2 \simeq \int d\omega\kappa(\omega)$ the Zeno time. As we have shown, there is a characteristic transition time τ^* [coupling K^*], such that one obtains:

$$\begin{aligned} &\text{for}\tau < \tau^* \ [K > K^*] \Rightarrow \text{decoherence suppression}: \gamma(\tau) < \gamma \ [\gamma(K) < \gamma], \\ &\text{for}\tau > \tau^* \ [K < K^*] \Rightarrow \text{decoherence enhancement}: \gamma(\tau) > \gamma \ [\gamma(K) > \gamma]. \end{aligned} \tag{52}$$

Therefore, in order to obtain a suppression of decoherence, the interruptions/coupling must be *very* frequent/strong. Notice that both τ^* and $2\pi/K^*$ are not simply related to the inverse bandwidth $2\pi W^{-1}$: they can be in general (much) shorter. For instance, in the Ohmic polynomial case (10), one gets

$$\begin{cases} \tau_Z^* \simeq 2\pi W^{-1}\left(2(n-1)\alpha_n^2\frac{\Omega}{W}\right) \ll 2\pi W^{-1}\,, \\ \tau_{\mathrm{k}}^* \simeq 2\pi W^{-1}\frac{\alpha_n}{2}\left(\frac{\alpha_n\pi^2}{4}\frac{\Omega}{W}\right)^{\frac{1}{2n-1}} \ll 2\pi W^{-1}\,, \\ K^* \simeq W\alpha_n^{-1}\left(\frac{2}{\alpha_n}\frac{W}{\Omega}\right)^{\frac{1}{2n-1}} \gg W\,, \end{cases} \tag{53}$$

where $\alpha_n = (\sqrt{\pi}/2)\Gamma(n-3/2)/\Gamma(n-1) \leq \pi/2$ is a coefficient of order 1 and n characterizes the polynomial fall off of the form factor (10). The above times/coupling may be (very) difficult to achieve in practice. In fact, we see here that the relevant timescale is not simply the inverse bandwidth $2\pi W^{-1}$, but can be much shorter if $\Omega \ll W$, as is typically the case. These

conclusions, summarized here for the simple case of a qubit, are valid in general, when one aims at protecting from decoherence an N-dimensional subspace.

Acknowledgments We thank D. Lidar, H. Nakazato, S. Tasaki and A. Tokuse for many discussions. This work is partly supported by the bilateral Italian-Japanese project 15C1 on "Quantum Information and Computation" of the Italian Ministry for Foreign Affairs.

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CLAUSER-HORNE INEQUALITY FOR ELECTRON COUNTING STATISTICS IN MULTITERMINAL MESOSCOPIC CONDUCTORS

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In this paper we derive the Clauser-Horne (CH) inequality for the full electron counting statistics in a mesoscopic multiterminal conductor and we discuss its properties. We first consider the idealized situation in which a flux of entangled electrons is generated by an *entangler*. Given a certain average number of incoming entangled electrons, the CH inequality can be evaluated for different numbers of transmitted particles. Strong violations occur when the number of transmitted charges on the two terminals is the same ($Q_1 = Q_2$), whereas no violation is found for $Q_1 \neq Q_2$. We then consider two actual setups that can be realized experimentally. The first one consists of a three terminal normal beam splitter and the second one of a hybrid superconducting structure. Interestingly, we find that the CH inequality is violated for the three terminal normal device. The maximum violation scales as $1/M$ and $1/M^2$ for the entangler and normal beam splitter, respectively, $2M$ being the average number of injected electrons. As expected, we find full violation of the CH inequality in the case of the superconducting system.

1. Introduction

Most of the work on entanglement has been performed in optical systems with photons [1], cavity QED systems [2] and ion traps [3]. Only recently attention has been devoted to the manipulation of entangled states in a solid state environment. This interest, originally motivated by the idea to realize a solid state quantum computer [4,5,6], has been rapidly growing and by now several works discuss how to generate, manipulate and detect entangled states in solid state systems. It is probably worth to emphasize already at this point that, differently from the situation encountered in quantum optics, in solid state system entanglement is rather common. What is not trivial is its control and detection (especially if the interaction between the different subsystems forming the entangled state is switched off).