Exponential Behavior of a Quantum System in a Macroscopic Medium

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An exponential behavior at all times is derived for a solvable dynamical model in the weak-coupling macroscopic limit. Some implications for the quantum measurement problems are discussed, in particular in connection with dissipation.

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"Decoherence" has become an important keyword in the quantum measurement theory [1,2]. Many people ascribe the measurement process to a dephasing process [2,3] and have tried to derive it as a consequence of the interaction with measuring devices, within the framework of quantum mechanics. Because decoherence technically means the elimination of the off-diagonal elements of the density matrix, a system described by such a diagonal density matrix should exhibit a purely stochastic behavior, and we naturally expect a close connection with a dissipative and irreversible behavior [4].

On the other hand, the temporal evolution of a quantum mechanical system, initially prepared in an eigenstate of the unperturbed Hamiltonian, is known to be roughly characterized by three distinct regions [5]: a Gaussian behavior at short times, a Breit-Wigner exponential decay at intermediate times, and a power law at long times. It is well known that the asymptotic dominance of the exponential behavior is representative of a purely stochastic evolution and can be derived quantum mechanically [6] in the weak-coupling macroscopic limit (the so-called van Hove's limit). One may expect a close connection between dissipation and exponential decay [4]. The Gaussian short-time behavior is of particular significance due, in particular, to the so-called quantum Zeno effect [7].

The aim of this Letter is to discuss the temporal behavior of a solvable dynamical model which describes the interaction between a macroscopic and a macroscopic system in the weak-coupling macroscopic limit. Some implications for the quantum measurement problem will also be analyzed, in particular, in connection with dissipation. We shall base our discussion on the AgBr model [8] that has played an important role in the quantum measurement problem and its modified version [9], which is able to take into account energy-exchange processes. Our exact calculation shows that the model realizes the so-called diagonal singularity [6] and can display the occurrence of an exponential regime *at all times* in the weak-coupling macroscopic limit.

The modified AgBr Hamiltonian [9] describes the interaction between an ultrarelativistic particle Q and a one-dimensional N-spin array (D system). The array is a caricature of a linear "photographic emulsion" of AgBr molecules, when one identifies the *down* state of the spin with the undivided molecule and the *up* state with the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential. The total Hamiltonian for the $Q + D$ system reads

$$H = H_0 + H', \qquad H_0 = H_Q + H_D, \tag{1}$$

where H_Q and H_D, the free Hamiltonians of the Q particle and of the "detector" D, respectively, and the interaction Hamiltonian H' are written as

$$H_Q = c\hat{p}, \qquad H_D = \frac{1}{2}\hbar\omega\sum_{n=1}^{N}(1+\sigma_3^{(n)}),$$

$$H' = \sum_{n=1}^{N} V(\hat{x}-x_n)\Big[\sigma_+^{(n)}\exp\Big(-i\frac{\omega}{c}\hat{x}\Big) + \sigma_-^{(n)}\exp\Big(+i\frac{\omega}{c}\hat{x}\Big)\Big], \tag{2}$$

where $\hat{p}$ is the momentum of the Q particle, $\hat{x}$ its position, V a real potential, x_n ($n = 1,\ldots,N$) the positions of the scatterers in the array ($x_n > x_{n-1}$), and $\sigma_{i,\pm}^{(n)}$ the Pauli matrices acting on the nth site. An interesting feature of the above Hamiltonian, as compared to the original one [8], is that we are not neglecting the energy H_D of the array, namely, the energy gap between the two states of each molecule. This enables us to take into account energy-exchange processes between Q and the spin system D. The original Hamiltonian [8] is reobtained in the $\omega = 0$ limit.

The temporal evolution of the system under investigation is best disclosed by studying the behavior of the propagator. Following van Hove's pioneering work [6], one could calculate the propagator perturbatively in this model. Observe that the interaction Hamiltonian H' has nonvanishing matrix elements only between those eigenstates of H_0 whose spin-quantum numbers differ by 1. It is important to note, in this connection, that the Q-particle state $|cp\rangle$, characterized by the energy cp, is changed by H' into the state $|cp \pm \hbar\omega\rangle$, if $\omega \neq 0$. We can therefore

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expect a dissipation effect and the appearance of the diagonal singularity, which leads to the master equation [6]. One could further develop the perturbative treatment just following Ref. [6]; however, the solvability of the present model enables us to perform a nonperturbative treatment and yields an exact expression for the propagator.

The evolution operator in the interaction picture

$$U(t,t') = e^{iH_0 t/\hbar} e^{-iH(t-t')/\hbar} e^{-iH_0 t'/\hbar} = e^{-i\int_{t'}^{t} H_I'(t'')\,dt''/\hbar}, \tag{3}$$

where $H_I'(t)$ is the interaction Hamiltonian in the interaction picture, can be computed exactly [9] as

$$U(t,t') = \prod_{n=1}^{N} \exp\left(-\frac{i}{\hbar}\int_{t'}^{t} V(\hat{x} + ct'' - x_n)\,dt'' \times \left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\hat{x}\right) + \text{H.c.}\right]\right). \tag{4}$$

In what follows, we set $t' = 0$ for notational simplicity. Define

$$\alpha_n \equiv \alpha_n(\hat{x}, t) \equiv \int_0^t V(\hat{x} + ct' - x_n)\,dt'/\hbar, \tag{5}$$

which can be viewed as a "tipping angle" of the nth spin if one identifies V with a magnetic field B [10], and

$$\sigma_\pm^{(n)}(\hat{x}) \equiv \sigma_\pm^{(n)} \exp\left(\mp i\frac{\omega}{c}\hat{x}\right), \tag{6}$$

which satisfy, together with $\sigma_3^{(n)}$, the SU(2) algebra

$$\left[\sigma_-^{(n)}(\hat{x}), \sigma_+^{(n)}(\hat{x})\right] = -\sigma_3^{(n)}, \qquad \left[\sigma_\pm^{(n)}(\hat{x}), -\sigma_3^{(n)}\right] = \pm 2\sigma_\pm^{(n)}(\hat{x}). \tag{7}$$

We can now return to the Schrödinger picture by inverting Eq. (3). We disentangle the exponential (4) by making use of Eq. (7) and obtain [11]

$$e^{-iHt/\hbar} = e^{-iH_0 t/\hbar} \prod_{n=1}^{N} \left(e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})} e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}} e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}\right). \tag{8}$$

Notice that the evolution operators (4) and (8) are expressed in a factorized form: This is a property of a rather general class of similar Hamiltonians [12].

We shall now concentrate our attention on the situation in which the Q particle is initially at the position $x' < x_1$, where x_1 is the position of the first scatterer in the linear array and is moving towards the array with speed c. The spin system is initially set in the ground state $|0\rangle_N$ of the free Hamiltonian H_D (all spins down). This choice of the ground state is meaningful from a physical point of view, because the Q particle is initially outside D.

The propagator, defined by

$$G(x,x',t) \equiv \langle x| \otimes {}_N\langle 0|e^{-iHt/\hbar}|0\rangle_N \otimes |x'\rangle, \tag{9}$$

is easily calculated from Eq. (8) and we obtain

$$\begin{aligned} G(x,x',t) &= \langle x| \otimes {}_N\langle 0|e^{-ic\hat{p}t/\hbar} \prod_{n=1}^{N} \left(e^{-\ln\{\cos[\alpha_n(\hat{x},t)]\}\sigma_3^{(n)}}\right)|0\rangle_N \otimes |x'\rangle \\ &= \langle x|x' + ct\rangle \prod_{n=1}^{N} \left(e^{\ln\{\cos[\alpha_n(x',t)]\}}\right) \\ &= \delta(x - x' - ct) \prod_{n=1}^{N} \cos\alpha_n(x',t). \end{aligned} \tag{10}$$

Observe that, due to the choice of the free Hamiltonian H_Q in Eq. (2), the Q wave packet does not disperse and moves with constant speed c. We place the spin array at the far right of the origin ($x_1 > 0$) and consider the case where potential V has a compact support and the Q particle is initially located at the origin $x' = 0$, i.e., well outside the potential region of D. The above equation shows that the evolution of Q occurs only along the path $x = ct$. Therefore we obtain

$$G(x,0,t) = \delta(x - ct) \prod_{n=1}^{N} \cos\tilde{\alpha}_n(t), \qquad \tilde{\alpha}_n(t) \equiv \int_0^{ct} V(y - x_n)\,dy/\hbar c. \tag{11}$$

This result is *exact.*

Let $V_0\Omega \equiv \int_{-\infty}^{\infty} V(x)\,dx$ and call q the "spin-flip" probability, i.e., the probability of dissociating one AgBr molecule,

$$q \equiv \sin^2\tilde{\alpha}_n(\infty) = \sin^2\left(\frac{V_0\Omega}{\hbar c}\right). \tag{12}$$

Note that $qN \equiv \bar{n}$ represents the average number of dissociated molecules. We shall now consider the weak-coupling macroscopic limit [9]

$$q \simeq \left(\frac{V_0\Omega}{\hbar c}\right)^2 = O(N^{-1}), \tag{13}$$

which is equivalent to the requirement that $\bar{n} = qN$ be finite. Notice that if we set

$$x_n = x_1 + (n-1)\Delta, \qquad L = x_N - x_1 = (N-1)\Delta, \tag{14}$$

the scaled variable $z_n \equiv x_n/L$ can be considered as a continuous one z in the above limit, for $\Delta/L \to 0$ as

$N \to \infty$. Therefore, a summation over n is to be replaced by a definite integration

$$q \sum_{n=1}^{N} f(x_n) \to q \frac{L}{\Delta} \int_{x_1/L}^{x_N/L} f(Lz)\, dz \simeq \overline{n} \int_{x_1/L}^{x_N/L} f(Lz)\, dz . \tag{15}$$

This type of integration gives a finite result if the function f is scale invariant, because the integration volume is considered to be finite from the physical point of view; in fact, the quantities x_1/L and x_N/L should be of the order of unity even in the $L \to \infty$ limit. It will be shown below [Eq. (16)] that in the present case the function f is indeed scale invariant.

For the sake of simplicity, we shall restrict our attention to the case of δ-shaped potentials, by setting $V(y) = (V_0\Omega)\delta(y)$. This hypothesis is in fact too restrictive: In the following, we shall see that the requirement that V has a compact support (local potentials) would suffice. We obtain

$$\begin{aligned} G &\propto \exp\left(\sum_{n=1}^{N} \ln\left\{\cos \int_{-x_n}^{ct-x_n} (V_0\Omega/\hbar c)\delta(y)\, dy\right\}\right) \\ &= \exp\left(\sum_{n=1}^{N} \ln\{\cos[(V_0\Omega/\hbar c)\theta(ct - x_n)]\}\right) \\ &\to \exp\left(-\frac{\overline{n}}{2} \int_{x_1/L}^{x_n/L} \theta(ct - Lz)\, dz\right) \\ &= \exp\left(-\frac{\overline{n}}{2}\left[\frac{ct - x_1}{L}\theta(x_N - ct)\right.\right. \\ &\qquad \left.\left. \times \theta(ct - x_1) + \theta(ct - x_N)\right]\right), \end{aligned} \tag{16}$$

where θ is the step function and the arrow denotes the weak-coupling macroscopic limit (13).

This brings about an exponential regime *as soon as the interaction starts:* Indeed, if $x_1 < ct < x_N$,

$$G \propto \exp\left(-\overline{n}\, \frac{c(t - t_0)}{2L}\right), \tag{17}$$

where $t_0 = x_1/c$ is the time at which the Q particle meets the first potential. Notice that there is *no* Gaussian behavior at short times and *no* power law at long times.

Observe that if $ct > x_N$ (that corresponds to the case in which Q has gone through D and the interaction is over), we have

$$G \propto e^{-\overline{n}/2}. \tag{18}$$

This could also be obtained directly from Eq. (11) and is in complete agreement with previous results [9], because $|G|^2$ is nothing but the probability that Q goes through the spin array *and* leaves it in the ground state.

It is well known [5], [13] that deviations from exponential behavior at short times are a consequence of the finiteness of the mean energy of the initial state. If the position eigenstates in Eq. (9) are substituted with wave packets of size a, a detailed calculation shows that the exponential regime is attained a short time after t_0, of the order of a/c, which, in the present model, can be made arbitrarily small. The region $t \sim t_0 + O(a/c)$ may be viewed as a possible residuum of the short-time Gaussian-like behavior. For this reason, the temporal behavior derived in this Letter is not in contradiction with some general theorems [5,7,13].

What causes the occurrence of the exponential behavior displayed by our model? Our analysis suggests that the exponential behavior is mainly due to the locality of the potentials V and the factorized form of the evolution operator U. As already stressed, the factorization is a general property of a class of similar Hamiltonians [12].

It is very interesting to bring to light the profound link between the weak-coupling macroscopic limit $qN = \overline{n} =$ finite considered in this Letter and van Hove's "$\lambda^2 T$" limit [6]. First, it is important to note that in the weak-coupling macroscopic limit van Hove's so-called diagonal singularity naturally appears in the present model. It is easy to check that for each diagonal matrix element of H'^2, there are N intermediate-state contributions: Indeed, for example,

$$\begin{aligned} &\langle 0, \ldots, 0|H'^2|0, \ldots, 0\rangle \\ &\qquad = \sum_{j=1}^{N} |\langle 0, \ldots, 0|H'|0, \ldots, 0, 1_{(j)}, 0, \ldots, 0\rangle|^2. \end{aligned} \tag{19}$$

On the other hand, at most two states can contribute to each off-diagonal matrix element of H'^2. This ensures that only the diagonal matrix elements are kept in the weak-coupling macroscopic limit, $N \to \infty$ with $qN < \infty$, which is the realization of diagonal singularity in our model. The link with the $\lambda^2 T$ limit is easily evinced by observing that q, in Eq. (13), is nothing but the square of a coupling constant (van Hove's λ) and that N ($\propto L$) can be considered proportional to the total interaction time T. Notice that the "lattice spacing" Δ, the inverse of which corresponds to a density in our one-dimensional model, can be kept finite in the limit. [In such a case, we have to express everything in terms of scaled variables, that is, $\tau \equiv t/L$, z_1 and z_N, introduced just after Eq. (14) and $\zeta \equiv a/L$, where a is the size of the wave packet.]

The role played by the energy gap ω need also be clarified. As a matter of fact, ω plays a very important role to guarantee the consistency of the physical framework: If $\omega = 0$, all spin states would be energetically degenerated, and the choice of the σ_3-diagonal representation would be quite *arbitrary*. In other words, a nonvanishing ω (or H_D) logically allows us to use the eigenstates of σ_3 in order to evaluate the relevant matrix elements. Though ω does not appear in our final result (17), it certainly does in other propagators like $\langle c\tilde{p}| \otimes {}_N\langle 0|e^{-iHt/\hbar}|0\rangle_N \otimes |c\tilde{p}'\rangle$, where $|c\tilde{p}\rangle$ stands for a wave packet.

Finally, let us comment on the connection between dissipation and exponential decay. Leggett [4], by discussing the role of the environment in connection with the collapse of the wave function, stressed the central relevance of the problem of dissipation to the quantum measurement theory. We find his argument very convincing. However, at the same time, it seems to us that this idea needs to be sharpened: The temporal behavior derived in this Letter, yielding an exponential "probability dissipation," is certainly related to dephasing (decoherence) effects of the same kind as those encountered in quantum measurements. The exchange of energy between the particle and the "environment" (our spin system) can be considered practically irreversible. On the other hand, in the present model, the final result (17) does *not* depend on ω. We do not want to draw any crucial conclusion on this issue, starting from the present analysis, on the basis of the particular model considered. We leave this as an open problem that is certainly worth further investigation.

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