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Solvable Dynamical Model for a Quantum Measurement Process

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A model Hamiltonian describing an energy-exchange process between an ultrarelativistic particle and a one-dimensional spin array is proposed and solved exactly. Interesting relations with the quantum measurement problem are discussed.

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Solvable models in quantum mechanics are very useful because they often provide physicists with a clear understanding of the physical phenomena involved. In particular, they are very helpful in order to comprehend the so-called quantum measurement problem [1,2]. A famous model Hamiltonian originally proposed by Hepp [3] is a good example of this sort, and has been extensively studied by several authors [4–8] because, in spite of its simplicity, it yields rich physical insights.

The above-mentioned Hamiltonian is usually referred to as the Coleman-Hepp or "AgBr" Hamiltonian, and describes the interaction between an ultrarelativistic particle Q and a one-dimensional N-spin array (D system). One can think, for instance, of a linear emulsion of AgBr molecules, the *down* state corresponding to the undivided molecule, and the *up* state corresponding to the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential.

The total Hamiltonian for the $Q+D$ system is

$$H=H_Q+H', \tag{1}$$

where H_Q is the free Hamiltonian of the particle and H' the interaction Hamiltonian. These are explicitly written as

$$H_Q=c\hat{p}\,,\qquad H'=\sum_{n=1}^{N} V(\hat{x}-x_n)\sigma_1^{(n)}\,, \tag{2}$$

where $\hat{p}$ is the momentum of the particle, $\hat{x}$ its position, V is a real potential, x_n ($n=1,\ldots,N$) are the positions of the scatterers in the array, $\sigma_1^{(n)}$ is the Pauli matrix acting on the nth site, and we make use of the caret only for the position and momentum operators.

This Hamiltonian is a nice model of a typical measurement process and can be solved exactly. However, we should remark that the above interaction Hamiltonian does not take into account the possibility of energy exchange between the particle and the spin system: The former never loses (or gains) energy as a consequence of the interaction. In other words, the energy levels of the spin system are completely neglected, even though the total energy is conserved. This is not really satisfactory if we want to regard the spin system as a detecting device, because we are implicitly assuming that we are able to distinguish *energetically* different states of the array: On the other hand, this can be done only via a free Hamiltonian of the spin system, which is absent in the above description.

In this Letter we shall improve this situation by taking into account both the free energy of the D system and the energy transfer between the Q and D systems. This will be accomplished by adding the free Hamiltonian of the spin array and by introducing an appropriate operator into the interaction Hamiltonian. These modifications will make the model more consistent and realistic. Remarkably, we shall see that the model will remain solvable if a "resonance condition," to be defined later, is met.

Let us start our considerations by writing the total

Hamiltonian for the $Q+D$ system as

$$H=H_0+H', \quad H_0=H_Q+H_D, \tag{3}$$

where H_Q and H_D are the free Hamiltonians of the particle and of the detector, respectively, and H' is the interaction Hamiltonian. These are written as

$$\begin{aligned} H_Q &= c\hat{p}, \\ H_D &= \frac{1}{2}\hbar\omega\sum_{n=1}^{N}(1+\sigma_3^{(n)}), \\ H' &= \sum_{n=1}^{N} V(\hat{x}-x_n)\sigma_1^{(n)}\exp\left[i\frac{\omega}{c}\sigma_3^{(n)}\hat{x}\right] \\ &= \sum_{n=1}^{N} V(\hat{x}-x_n)\left[\sigma_+^{(n)}\exp\left[-i\frac{\omega}{c}\hat{x}\right]+\sigma_-^{(n)}\exp\left[+i\frac{\omega}{c}\hat{x}\right]\right], \end{aligned} \tag{4}$$

where the notation is the same as above, $\sigma_3^{(n)}$ is the Pauli matrix acting on the nth site, and $\sigma_\pm^{(n)}=(\sigma_1^{(n)}\pm i\sigma_2^{(n)})/2$. Notice that the energy difference between the two states of the molecule is $\hbar\omega$, and that the previous Hamiltonian [Eq. (2)] is reobtained in the $\omega\to 0$ limit.

In contrast with the previous analyses [3–8], we are not neglecting the free energy of the scatterers, represented by H_D, and are taking into account the energy exchange between the Q particle and the spin system. This is automatically accomplished by the interaction Hamiltonian, whose action can be decomposed in the following way:

$$\begin{aligned} H'_{(n)}|p,\downarrow_{(n)}\rangle &= V(\hat{x}-x_n)|p-\hbar\omega/c,\uparrow_{(n)}\rangle, \\ H'_{(n)}|p,\uparrow_{(n)}\rangle &= V(\hat{x}-x_n)|p+\hbar\omega/c,\downarrow_{(n)}\rangle, \end{aligned} \tag{5}$$

where $H'_{(n)}$ is the H' term acting on the nth site, $|p,\downarrow_{(n)}\rangle$ represents a state in which the Q particle has momentum p and the nth molecule is undivided (spin down), and analogously for the other cases. We understand from Eq. (5) that the interaction Hamiltonian H' satisfies a "resonance condition," because the energy acquired or lost by the Q particle in every single interaction matches exactly the energy required to provoke one spin flipping.

We shall work in the interaction picture. The evolution operator

$$U(t,t')=e^{iH_0t/\hbar}e^{-iH(t-t')/\hbar}e^{-iH_0t'/\hbar} \tag{6}$$

satisfies

$$i\hbar\frac{dU(t,t')}{dt}=H'_I(t)U(t,t'), \quad \text{with } U(t',t')=1, \tag{7}$$

and $H'_I(t)$, the interaction Hamiltonian in the interaction picture, can be computed exactly as

$$\begin{aligned} H'_I(t) &= e^{iH_0t/\hbar}H'e^{-iH_0t/\hbar} \\ &= \sum_{n=1}^{N} V(\hat{x}+ct-x_n)\sigma_1^{(n)}\exp\left[i\frac{\omega}{c}\sigma_3^{(n)}\hat{x}\right] \\ &= \sum_{n=1}^{N} V(\hat{x}+ct-x_n)\left[\sigma_+^{(n)}\exp\left[-i\frac{\omega}{c}\hat{x}\right]+\sigma_-^{(n)}\exp\left[+i\frac{\omega}{c}\hat{x}\right]\right]. \end{aligned} \tag{8}$$

[Notice that in the interaction picture the operator $\hat{x}$ is shifted by ct, while $\sigma_\pm^{(n)}$ is multiplied by $\exp(\pm i\omega t)$, so that $\sigma_\pm^{(n)}\exp(\mp i\omega\hat{x}/c)$ remains unchanged.] Observe that

$$[H'_I(t),H'_I(t')]=0, \tag{9}$$

so that the solution to Eq. (7) is

$$U(t,t')=\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}H'_I(t'')dt''\right], \tag{10}$$

and a straightforward calculation yields the following S matrix:

$$S^{[N]}=\lim_{\substack{t\to\infty\\ t'\to-\infty}}U(t,t')=\prod_{n=1}^{N}S_{(n)}, \tag{11}$$

where

$$\begin{aligned} S_{(n)} &= \exp\left[-i\frac{V_0\delta}{\hbar c}\sigma^{(n)}\cdot\mathbf{u}\right] \\ &= \cos\left[\frac{V_0\delta}{\hbar c}\right]-i\sigma^{(n)}\cdot\mathbf{u}\sin\left[\frac{V_0\delta}{\hbar c}\right], \\ \mathbf{u} &= \left[\cos\left[\frac{\omega}{c}x\right],\sin\left[\frac{\omega}{c}x\right],0\right], \end{aligned} \tag{12}$$

and we have written $V_0\delta=\int_{-\infty}^{\infty}V(x)\,dx$. This allows us to define the "spin-flip" probability, i.e., the probability of dissociating one AgBr molecule, as

$$q=\sin^2\left(V_0\delta/\hbar c\right). \tag{13}$$

The initial D state is taken to be the ground state $|0\rangle_N$ (N spins down), and we shall first consider the situation in which the initial Q state is a plane wave. The evolution is easily computed by Eqs. (11)–(13) as

$$S^{[N]}|p,0\rangle_N=\sum_{j=0}^{N}\binom{N}{j}^{1/2}(-i\sqrt{q})^j(\sqrt{1-q})^{N-j}\left|p-j\frac{\hbar\omega}{c},j\right\rangle_N, \tag{14}$$

where we have used the notation $|p_j,j\rangle_N=|p_j\rangle|j\rangle_N$, with $p_j=p-j\hbar\omega/c$, $|j\rangle_N$ being the (symmetrized) state of D in which j molecules are excited.

Let us analyze now a typical interference experiment in which a divider splits an incoming wave function ψ into two branch waves ψ_1 and ψ_2, so that the initial state of the $Q+D$ system is

$$\Psi_I=(\psi_1+\psi_2)|0\rangle_N, \tag{15}$$

where $|\psi_i\rangle=\int dp_i\, c(p_i)|p_i\rangle$ ($i=1,2$) are one-dimensional wave packets, normalized to unity, and assume that only ψ_2 interacts with D. The final state of the total system is

$$\Psi_F=|\psi_1\rangle|0\rangle_N+S^{[N]}|\psi_2\rangle|0\rangle_N, \tag{16}$$

and after recombination of the two branch waves the probability of observing the particle is

$$P=|\Psi_F|^2, \tag{17}$$

Interference is observed when a phase shifter is inserted in one of the two paths, and the visibility of the interference pattern is readily calculated by Eqs. (14) and (16) as

$$\mathcal{V}=\frac{P_{\max}-P_{\min}}{P_{\max}+P_{\min}}={}_N\langle 0|S^{(N)}|0\rangle_N=(1-q)^{N/2}. \tag{18}$$

It is also interesting to compute the energy "stored" in the array after the interaction with the particle, as well as the fluctuation around the average:

$$\begin{aligned}&\langle H_D\rangle_F={}_N\langle 0|S^{[N]\dagger}H_DS^{[N]}|0\rangle_N=qN\hbar\omega,\\ &\langle\delta H_D\rangle_F=[\langle(H_D-\langle H_D\rangle_F)^2\rangle_F]^{1/2}=\sqrt{pqN}\,\hbar\omega,\\ &\frac{\langle\delta H_D\rangle_F}{\langle H_D\rangle_F}=\left(\frac{p}{qN}\right)^{1/2},\end{aligned} \tag{19}$$

where F stands for final state, $p=1-q$, and the trivial trace over the Q-particle states is suppressed.

Equations (12), (14), (18), and (19) are our main results. Observe that no approximation has been made in order to derive them: The result is exact and holds true for every value of N. We stress also that the quantities in Eq. (19) could not be calculated starting from the original Coleman-Hepp Hamiltonian (2), due to the absence of the free detector Hamiltonian H_D. The $N\to\infty$ limit is a somewhat delicate problem, especially in the case $\omega\neq 0$.

As was to be expected, for finite q, the interference pattern disappears in the N-infinite limit. This is essentially the case considered by Hepp [3] and Bell [4]. On the other hand, it is very interesting to consider the limit $N\to\infty$, $qN=\bar{n}=$finite [7]. In this case, the above quantities become

$$\begin{aligned}&\mathcal{V}\to e^{-\bar{n}/2},\quad \langle H_D\rangle_F\to\bar{n}\hbar\omega,\\ &\langle\delta H_D\rangle_F\to\bar{n}^{1/2}\hbar\omega,\quad \frac{\langle\delta H_D\rangle_F}{\langle H_D\rangle_F}\to\frac{1}{\bar{n}^{1/2}}.\end{aligned} \tag{20}$$

Notice that the simple $N\to\infty$ limit, with q finite and $q\neq 0$, yields only divergent or vanishing quantities. In the $N\to\infty$ limit, the quantity q can be shown to coincide numerically with the superselection charge [9] of the so-called many-Hilbert-space approach to the quantum measurement problem [10].

It is also worth stressing that $qN=\bar{n}$ represents the average number of excited molecules, so that interference and relative energy fluctuations "gradually" disappear as $\bar{n}$ increases [see Eq. (20)]. Observe also that for $\omega=0$ we recover every result previously obtained: In particular, the state in Eq. (14) is a generalized coherent state [6,8], and becomes a Glauber coherent state in the $N\to\infty$, $qN=$finite limit [7].

The next step would be to consider the $N\to\infty$ limit in the general $\omega\neq 0$ case. This problem is now being investigated [11], and there are interesting connections with another famous solvable Hamiltonian [12]. The links with another well-known model [13] in the $\omega=0$ case were already stressed in Ref. [8].

Finally, we would like to emphasize once more that the Hamiltonian we proposed in Eqs. (3) and (4) has a direct physical meaning and yields nice physical insights into the problem investigated. It would even be possible to generalize the results obtained to the nonrelativistic Q-particle case and to the off-resonance situation. In such cases, the model would remain solvable within reasonable approximations. We preferred to limit our attention to the case considered in this Letter because it avoids unnecessary complications.

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