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Hindered decay of an unstable system: A quantum Zeno effect

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Abstract. A direct test of the so-called "quantum Zeno effect" is proposed for a truly decaying system. It is suggested that the lifetime of an unstable atom can be extended by illuminating it with an intense laser beam at the frequency of another of its transitions. The "Zeno" time is also compared to the lifetime.

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1. Introduction

The quantum Zeno effect is a straightforward consequence of a very general property of the Schrödinger equation and of the measurement postulate (von Neumann's projection rule [1]): If the state of a quantum system evolves under the action of its Hamiltonian, in a small time δt its phase changes by $O(\delta t)$, while the norm of its projection onto the initial state changes by $O(\delta t^2)$. Therefore, by performing N measurements during a fixed time interval T, the probability of observing the system in the initial state is given by N/N^2, which becomes small as N is increased: In the limit of very frequent measurements, the state of the system does not change [2].

This effect (originally proposed for unstable systems) has been experimentally observed on an oscillating system [3,4]. Many physicists believed that the experiment was a clear evidence in support of the Copenhagen interpretation – von Neumann's projection rule. However, the subsequent lively discussion [5,6] clarified that the experimental data can be explained *without* making use of projection operators: A purely dynamical analysis, fully based on the Schrödinger equation, yields the very same results. The first (to our knowledge) who shed light on this important point was Peres [7], one decade before the above-mentioned experiment.

In the present paper we wish to discuss the quantum Zeno effect (QZE), propose an experimental test on a truly decaying system and finally investigate the time scales involved.

2. Short-time behaviour and quantum Zeno effect

Let us outline the main differences between the "classical" and the quantum mechanical evolution laws. In classical physics one heuristically assumes that the decay probability per unit time is a constant Γ (the inverse of the "lifetime" τ_{E}), that is characteristic of the system investigated and does not depend on other quantities, such as the total number N of unstable systems, their past history and the environment surrounding them. Let the (very large) number of systems at time t be $N(t)$; the number of systems that will decay in the time interval dt is

$$-dN = N\Gamma dt = N dt/\tau_{\mathrm{E}}, \tag{1}$$

which yields

$$N(t) = N_0 e^{-t/\tau_{\mathrm{E}}}, \tag{2}$$

where $N_0 = N(0)$. The "survival" or "nondecay probability" reads

$$P_{\mathrm{cl}}(t) = \frac{N(t)}{N_0} = e^{-t/\tau_{\mathrm{E}}} \simeq 1 - t/\tau_{\mathrm{E}} + \cdots, \tag{3}$$

where the expansion holds at short ($t \ll \tau_{\mathrm{E}}$) times. The above derivation is usually found in elementary physics textbooks. However, the assumptions underpinning it are delicate, for they reflect the basic features of a Markoffian process, in which memory and/or collective effects are absent.

What about quantum mechanics? Let $|a\rangle$ be the wave function of a given quantum system at time $t = 0$. The evolution is governed by the unitary operator $U(t) = \exp(-iHt/\hbar)$, where H is the Hamiltonian. The seminal work by Gamow [8] on the exponential decay law, as well as its derivation by Weisskopf and Wigner [9] are based on the assumption that a pole near the real axis of the complex energy plane dominates the temporal evolution of the quantum system. This assumption leads to a spectrum of the Breit–Wigner type [10] and to the Fermi Golden Rule [11], which yields the following expression for the lifetime

$$\tau_{\mathrm{E}}^{-1} = \frac{2\pi}{\hbar}\rho(E_a)|\langle a'|H|a\rangle|^2, \tag{4}$$

where $|a'\rangle$ is the final state, E_a the energy of the initial (as well as the final) state and $\rho(E_a)$ the density of states at the energy of the initial state. (We are neglecting, for simplicity, other quantum numbers yielding additional degeneracies.)

However, it was soon understood that a purely exponential decay law can neither be expected for very short [12] nor for very long [13] times. It became clear that the domain of validity of the exponential law is limited: the long-time power tails and the short-time quadratic behaviour are unavoidable consequences of the Paley–Wiener theorem [14] on Fourier transforms and of very general mathematical properties of the Schrödinger equation, respectively. The temporal behaviour of quantum mechanical systems is reviewed in [15].

Let us show how the short time behaviour entails the QZE. Define the survival or nondecay probability at time t as the square modulus of the survival amplitude

$$P(t) = |\langle a|e^{-iHt/\hbar}|a\rangle|^2. \tag{5}$$

By assuming that $|a\rangle$ is normalizable and belongs to the domain of definition of H (so that all moments of H in the state $|a\rangle$ are finite) one gets the short-time expansion

$$P(t) = 1 - t^2/\tau_Z^2 + \cdots \tag{6}$$

$$\tau_Z^{-1} \equiv \Delta H/\hbar = (1/\hbar)\left(\langle a|H^2|a\rangle - \langle a|H|a\rangle^2\right)^{1/2}, \tag{7}$$

which is quadratic in t and therefore yields a vanishing decay rate for $t \to 0$. [It is assumed that ΔH is nonvanishing, or, in other words, that $|a\rangle$ is not an eigenstate of H. Otherwise one simply gets $P(t) = 1$.] The short-time quadratic behaviour is in manifest contradiction with the exponential law (3), that predicts an initial nonvanishing decay rate τ_E^{-1}. We shall refer to the quantity τ_Z as to the "Zeno time," for reasons that will soon become apparent.

The quantum mechanical vanishing decay rate at short times can be exploited in order to slow down the decay process. Suppose we perform N measurements at equal time intervals $\tau = T/N$, in order to ascertain whether the system is still in its initial state. After each measurement, the system is "projected" on to the quantum mechanical state representing the result of the measurement and the evolution starts anew with a vanishing decay rate, according to (6). The probability of observing the initial state at the final time $T = N\tau$, after having performed the above-mentioned N measurements, reads

$$P^{(N)}(T) = [P(\tau)]^N = [P(T/N)]^N \simeq \left(1 - \frac{1}{\tau_Z^2}\left(\frac{T}{N}\right)^2\right)^N \tag{8}$$

$$\stackrel{N \text{ large}}{\sim} e^{-T^2/\tau_Z^2 N} \stackrel{N\to\infty}{\longrightarrow} \mathcal{P}(T) = 1. \tag{9}$$

Notice that T is finite, in the above. The conclusion (9) was named "quantum Zeno paradox" by Misra and Sudarshan [2]. Zeno was the Eleatic philosopher famous for his paradoxical arguments against the philosophical notion of "becoming:" His speed arrows would never reach their target, if closely looked at. It is remarkable that von Neumann had realized already in 1932 (ref. [1] p. 195 [p. 366, English translation]), that a quantum mechanical state can be "steered" into other states by a series of measurements in rapid succession, so that, if the final state coincides with the initial one, the evolution of a quantum system can be halted (or at least dramatically slowed down).

There are several reasons why (9) is to be regarded as a paradoxical result. First of all, the $N \to \infty$ limit is practically unattainable, in a real experiment. More important, such a limit is unphysical, for it is in contradiction with Heisenberg's uncertainty principle [16].

3. Hindered decay

The original ideas on the QZE involved truly *unstable* systems [2], rather than systems undergoing oscillations of some sort [3,4]. More to this, one should prevent repopulation effects that affect the experiments performed on oscillating systems [17]. It is therefore interesting to study the properties of an atomic system that is initially prepared in an excited state and undergoes a "measurement" of some sort.

Let us be more specific, and consider the following situation [18]: Prepare a 3-level atom in the excited level #2. The atom, if left undisturbed, would naturally decay to its ground state (assumed to be level #1) according to an (approximately) exponential law.

Suppose now that you shine on the atom a very intense laser beam whose frequency is approximately equal to $E_3 - E_1$. The atom, being continuously "monitored" by the laser beam, undergoes a sort of "continuous observation," in the following sense: If the electron makes a transition from level #2 to level #1, it is rapidly "snatched" away to level #3, so that any evidence that level #3 is populated would mean that level #2 has decayed. Level #2 is therefore continuously "observed" and we intuitively expect its decay to be hindered. [Notice that, in principle, it is *not* necessary that an experimenter *observes* the atom in level #3 (e.g. by detecting a photon spontaneously emitted by level #3): For all practical and computational purposes, it is enough to apply von Neumann's projection rules.] It is interesting to notice that Peres arrived at a similar conclusion, by writing [7]: "If the decay products are quickly removed by some external mechanism, [the survival amplitude] will remain constant."

A detailed calculation, following the above idea, has been presented in ref. [18]. Here, we shall only sketch the main conclusions. Consider the Hamiltonian ($\hbar = 1$)

$$
\begin{aligned}
H = & \sum_{j=1}^{3} E_j |j\rangle\langle j| + \sum_k \omega_k a_k^\dagger a_k \\
& + \sum_k a_k^\dagger (\phi_k |1\rangle\langle 2| + \Phi_k |1\rangle\langle 3|) + \text{H.c.},
\end{aligned} \tag{10}
$$

where $|j\rangle$ ($j = 1, 2, 3$) is the atomic state, the first term the free Hamiltonian of the 3-level atom, the second term the free Hamiltonian of the EM field and the third term the interaction Hamiltonian describing, in the rotating wave approximation, the $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ transitions. The c-numbers ϕ_k, Φ_k are the matrix elements for the atomic transitions.

We prepare our system in the initial state $|\Psi_0\rangle = |2\rangle \otimes |0_k, N_0\rangle$ (atom in level #2, no photons of energy $\omega_k \approx E_2 - E_1$ and $N_0 (\gg 1)$ photons of energy $\Omega_0 = E_3 - E_1$) and solve the time-dependent Schrödinger equation $i\partial\Psi_t/\partial t = H\Psi_t$. In the continuum limit, the survival amplitude can be expressed as an inverse Laplace transform

$$
\langle \Psi_0 | \Psi_t \rangle_B = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{st}}{s + Q(s)} ds, \tag{11}
$$

$$
Q(s) = \int dE \rho(E) |\varphi(E)|^2 \frac{s + iE}{(s + iE)^2 + B^2}, \tag{12}
$$

where $B^2 = N_0 |\Phi_0|^2$, $\rho(E)$ is the density of states at energy E and $\varphi(E)$ are the scaled matrix elements of the $1 \leftrightarrow 2$ transition. The evolution is dominated by the poles of the integrand, namely the zeroes of the equation $s + Q(s) = 0$. In turn, $Q(s)$ depends on the poles of its integrand in the complex E-plane.

Let us first consider the case $B = 0$ (no "measurement" field is applied). The denominator in (12) has one zero $E = is$, yielding

$$
\langle \Psi_0 | \Psi_t \rangle_{B=0} \simeq e^{(-\gamma/2 + i\triangle E)t}, \tag{13}
$$

$$
\triangle E = P \int dE \frac{g(E)}{E}, \quad \gamma = 2\pi\rho(E_0)|\varphi(E_0)|^2, \tag{14}
$$

where E_0 is the energy of the initial state Ψ_0. This is nothing but the Fermi Golden Rule [11]. Consider now the case $B \neq 0$. The denominator in (12) has two zeroes $E = \pm B + is$, yielding

$$\langle \Psi_0|\Psi_t\rangle_{B\neq 0} \simeq e^{(-\gamma'/2+i\triangle E')t}, \tag{15}$$

$$\begin{aligned}\triangle E' &= P\int dEg(E)\frac{E}{E^2-B^2},\\ \gamma' &= \pi\left[\rho(E_0+B)|\varphi(E_0+B)|^2+\rho(E_0-B)|\varphi(E_0-B)|^2\right].\end{aligned} \tag{16}$$

The presence of the B-field (which is associated with the $1 \leftrightarrow 3$ transition) modifies the lifetime of level #2 ! In particular,

$$\frac{\gamma}{\gamma'} = \frac{2\rho(E_0)|\varphi(E_0)|^2}{\rho(E_0+B)|\varphi(E_0+B)|^2+\rho(E_0-B)|\varphi(E_0-B)|^2}. \tag{17}$$

A quantitative estimate of the above ratio requires evaluation of the matrix elements φ and the phase space factor ρ. However, it is natural to expect that $\gamma' \ll \gamma$, as B becomes large: The decay is "hindered" by the presence of the B-field.

This proposal is a direct test of QZE on a truly decaying system and prevents repopulation effects of the type described in [17]. It is interesting to note that we are increasing the lifetime of an excited atom *without* exploiting the short-time nonexponential behaviour discussed in § 2. By contrast, our atom simply settles into a slower exponential: It is just the change in the lifetime that displays the new features of the evolution. This temporal evolution is "dominated" by the presence of the EM ("B") field [19,18].

4. Time scales

The lifetime $\tau_{\rm E}$ and the Zeno time $\tau_{\rm Z}$ depend on the structure of the Hamiltonian, and their comparison can be a difficult task. For example, several authors discussed the possibility that the proton decay has never been observed because the Zeno time $\tau_{\rm Z}^{\rm prot}$ might be longer than the lifetime of the Universe [20]. Note also that the Weisskopf–Wigner method does not provide any estimate for the validity of the approximations leading to the exponential law. It is therefore interesting to understand if and when the vanishing derivative at short times (6) may lead to observable consequences [21,7].

Suppose that the Hamiltonian can be written as

$$H = H_0 + H', \tag{18}$$

where H_0 and H' are the free and interaction Hamiltonians, respectively. Let $\{|r\rangle\}$ be a complete orthonormal set made up of the eigenvectors of H_0

$$H_0|r\rangle = E_r|r\rangle\ . \tag{19}$$

Let us make the "usual" assumption that the initial state $|a\rangle$ be one of the $|r\rangle$s. (This hypothesis can be dispensed with and, anyway, does not imply any lack of generality. It is worth noting that such an assumption is valid in the cases [3,4], [6] and [18].) The Golden Rule (4) becomes

$$\tau_{\rm E}^{-1} = \frac{2\pi}{\hbar}\rho(E_a)|\langle a'|H'|a\rangle|^2\ , \tag{20}$$

due to the orthogonality of $|a'\rangle$ and $|a\rangle$. Moreover,

$$\langle a|H|a\rangle = E_a + h_1, \qquad (h_1 \equiv \langle a|H'|a\rangle) \tag{21}$$

$$\langle a|H^2|a\rangle = E_a^2 + 2E_a h_1 + h_2^2, \qquad (h_2^2 \equiv \langle a|H'^2|a\rangle) \tag{22}$$

so that the Zeno time, defined in eq. (7), reads

$$\tau_Z^{-2} = (\triangle H)^2/\hbar^2 = \left(h_2^2 - h_1^2\right)/\hbar^2 = (\triangle H')^2/\hbar^2. \tag{23}$$

Assume also that

$$\langle r|H'|r\rangle = 0\,. \tag{24}$$

Note that this condition enables us to solve the Lippmann–Schwinger equation and corresponds to mass renormalization in field theory. Under this assumption, $h_1 = 0$ and, after a short manipulation, upon insertion of a complete set of states, one gets

$$\tau_Z^{-2} = \int dE_r \rho(E_r)|\langle r|H'|a\rangle|^2/\hbar^2, \tag{25}$$

where the density of states includes all available states at energy E_r and the matrix element is assumed independent of other quantum numbers. We see that the Zeno time involves an off-shell contribution. By contrast, the lifetime (20) involves only on-shell states, at the same energy as the initial state. In conclusion,

$$\frac{\tau_E}{\tau_Z^2} = \int \frac{dE_r}{2\pi\hbar} \frac{\rho(E_r)|\langle r|H'|a\rangle|^2}{\rho(E_a)|\langle a'|H'|a\rangle|^2} \equiv \frac{\mathcal{E}_0}{\hbar} \equiv \frac{1}{\tau_0} \tag{26}$$

The quantities τ_0 and $\mathcal{E}_0$ depend on the model investigated and on the very structure of the interaction Hamiltonian. Note that the integrand in (26) gets its most important contribution from the region $E_r \sim E_a$ and it is reasonable to assume that it (quickly) vanishes as E_r moves away from E_a. In effect, it "measures the ability of the [interaction] Hamiltonian to move the system away from its initial state" [21]. We might interpret τ_0 as the time spent by the system to "explore" other states besides the final state $|a'\rangle$; $\mathcal{E}_0$ is an estimate of the energy involved in such a process, according to Heisenberg's uncertainty relation. It is interesting that the Zeno time can be viewed as a geometric mean $\tau_Z^2 = \tau_E \tau_0$. There is hopefully more to come.

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