Loss of quantum-mechanical coherence in a measurement process

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A quantum-mechanical measurement process is analyzed in terms of a model Hamiltonian describing the interaction between a quantum system (a "particle") and a macroscopic apparatus (a "detector"), which is assumed to be made up of N two-level elementary constituents ("molecules"). The description of the molecule locations introduces an effective fluctuating coupling constant, and this provokes a loss of quantum-mechanical coherence in the limit of large N. It is argued that coherence is lost *statistically*, as a result of the interaction: The collapse of the wave function is indeed obtained when the same experiment is performed many times, as a result of the microscopic differences among macroscopically identical initial states of the detector. In this way, insight is obtained into the mechanism engendering the loss of coherence suffered by a quantum-mechanical system when interacting with a macroscopic apparatus, and the concept of "wave-function collapse" is replaced by that of a *statistically defined* dephasing process. No classical behavior of the detection system is postulated and the presence of no external observer is required.

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I. INTRODUCTION

The problem of understanding whether a quantum-mechanical measurement process can be analyzed within the quantum-mechanical formalism has been a source of discussion for some 60 years. von Neumann's projection rules [1] are indeed to be added to the quantum formalism in order to account for the transition from a pure to a mixed state, and this makes quantum mechanics a non-self-contained theory.

Many attempts have been made in order to solve this dilemma. The Copenhagen school suggested that the transition from a pure to a mixed state (the so-called *wave-function collapse)* cannot be described within quantum mechanics, and therefore an external "classical" apparatus is required in order to provoke the collapse. Some eminent scientists have even gone so far as to state that the classical apparatus must comprise a human being, opening the door to subjectivistic interpretations of the quantum formalism. Other physicists have proposed to modify the Schrödinger equation by adding nonlinear terms, but this has the consequence of questioning the applicability of the superposition principle, whose validity has been beautifully checked in many different physical experiments. Other noteworthy attempts to solve the problem have dealt mainly with the introduction of superselection rules in the Hilbert space describing the measuring apparatus, or with the assumption that the Schrödinger equation describes a kind of averaged motion over an underlying, as yet unknown, stochastic process. In these last two cases the problem can be consistently solved, but at the cost of introducing the necessary ingredients in the very premises or of sacrificing locality, respectively. For a good review of these as well as many other interesting proposed solutions, see Ref. [2].

In this paper we shall give a concrete example of interaction between an elementary quantum system and a model detector. On the basis of this example, it will be argued that coherence is lost statistically, when the experiment is performed many times. In this sense, we believe that the concept of "wave-function collapse" is rather to be replaced by that of a *statistically defined dephasing process,* when the results relative to many events are accumulated. We shall endeavor to refrain from drawing very general conclusions on the basis of our final results: Indeed, the general problem requires much care, since we are still far from a completely satisfactory solution.

Our attention will be focused on two approaches that have been recently put forward by Cini [3] and Machida and Namiki [4], respectively. We have already tried to "blend" these two approaches in a previous paper [5], and our analysis has shown that it is indeed possible to explain the evolution from a pure to a mixed state without resorting to intrinsically classical apparatuses. In the present paper, we shall follow a more consistent approach, without limiting our analysis to a perturbative expansion.

The approaches by Cini and Machida and Namiki are, in a certain sense, both similar and discordant. They are similar, because both describe the measurement process and the measuring apparatus (the detector) *within* quantum mechanics, and do not postulate a "classical" behavior. They are discordant because the former ascribes the evolution from a pure to a mixed state to the second law of thermodynamics, while the latter explains this evolution via a "many-Hilbert-space" (MHS) structure of the (macroscopic) detector, by means of a superselection rule. In Secs. II and III we will show that it is possible to rederive Cini's results in terms of an alternative Hamiltonian, which was independently studied by several authors [6–8]. In Sec. IV a phase transition, previously proposed by Cini in a somewhat different context, will be analyzed in the light of our alternative proposal. In Sec. V we will put forward a consistent interpretation of our results in terms of the MHS theory; at the same time a *statistical* approach will be proposed.

Needless to say, the present paper will not provide "the solution" to the philosophical enigmas of quantum theory. Section VI will focus on the limits of our analysis and our model, and discuss critically our results.

II. THE PARTICLE-DETECTOR SYSTEM

Consider a two-level quantum system Q, described by the vector

$$\chi_0=(c_+u_+ + c_-u_-)\phi(\mathbf{r}) , \tag{2.1}$$

where $|c_+|^2+|c_-|^2=1$, $u_\pm$ are the eigenstates of τ_3, the third Pauli matrix, and $\phi(\mathbf{r})$ is a wave packet, normalized to 1. Let Q undergo a spectral decomposition, i.e., an interaction after which spatially separated wave packets correspond to different eigenvectors of the observable to be measured, so that the wave function becomes

$$\chi=c_+u_+\phi_+(r)+c_-u_-\phi_-(\mathbf{r}) , \tag{2.2}$$

where $\phi_\pm$ are distinct wave packets, traveling in different regions of space. After the spectral decomposition, one of the two branch waves of Q interacts with a detector D, which is assumed to be made up of N two-level elementary constituents (molecules); the molecules are identical, and the energy difference between the ground and excited state is $\hbar\omega$. The total Hamiltonian for the $Q+D$ system is written as

$$H=H_Q+H_D+H' , \tag{2.3}$$

where H_Q is the free Hamiltonian of the system Q, H_D the free Hamiltonian of the detector D, and H' the interaction Hamiltonian. These are explicitly written as

$$\begin{aligned} H_Q&=\mathbf{p}^2/2m , \\ H_D&=\tfrac{1}{2}\hbar\omega\sum_{n=1}^{N}(1-\sigma_3^{(n)}) , \\ H'&=\tfrac{1}{2}(1+\tau_3)\sum_{n=1}^{N}V(\mathbf{r}-\mathbf{r}_n)\sigma_1^{(n)} , \end{aligned} \tag{2.4}$$

where $\mathbf{p}$ is the momentum of the particle, $\mathbf{r}$ its position, V is a real potential, $\mathbf{r}_n$ $(n=1,\ldots,N)$ are the positions of the elementary constituents of D, and $\sigma_1^{(n)},\sigma_3^{(n)}$ are Pauli matrices acting on the nth constituent; for instance, we can think of a detector made up of AgBr molecules. The up state then corresponds to the undivided molecule, and the down state to the dissociated molecule (Ag and Br atoms). The term $\frac{1}{2}(1+\tau_3)$ allows us to limit our attention to a two-channel space representation for the system Q, in such a way that only the u_+ component of χ in Eq. (2.2) yields a nonvanishing interaction.

Different variants of the "AgBr" Hamiltonian H have been studied independently by several authors [6–8]. In contrast with the previous analyses, we will consider the three-dimensional case, in which $\mathbf{r},\mathbf{r}_n\in\mathbb{R}^3$ [the previous models always dealt with a *linear* array of molecules, in which $(\mathbf{r}-\mathbf{r}_n)\to(x-na)$, with a constant]. In Ref. [5], it has been shown that, under general conditions, the Hamiltonian H is equivalent to the one studied in Ref. [3]. Let us briefly review the most salient points of our previous analysis.

Assume that $H'\gg H_Q,H_D$, which corresponds approximately to the case of vanishing energy difference between the ground and ionized states of the detector's constituents, and of the Q particle being at rest in the detector, for a certain "interaction time" to be defined later; in Yang's effective potential approximation, we have

$$V(\mathbf{r})=V_0\delta\Omega\delta^3(\mathbf{r}), \quad \int V(\mathbf{r})d^3\mathbf{r}=V_0\delta\Omega , \tag{2.5}$$

where V_0 and $\delta\Omega$ are the potential "strength" and "volume," respectively. Let us define now a density of particles per unit volume

$$\rho(\mathbf{r})\equiv\sum_{n=1}^{N}\delta^2(\mathbf{r}-\mathbf{r}_n), \quad \int d^3\mathbf{r}\,\rho(\mathbf{r})=N , \tag{2.6}$$

and "smear out" the molecules inside the detector, by means of the substitution

$$\delta^3(\mathbf{r}-\mathbf{r}_n)\to\rho(\mathbf{r})/N \quad \forall n=1,\ldots,N . \tag{2.7}$$

The Hamiltonian becomes

$$H\simeq H'=\tfrac{1}{2}(1+\tau_3)\frac{V_0\delta\Omega}{N}\rho(\mathbf{r})\sum_{n=1}^{N}\sigma_1^{(n)} . \tag{2.8}$$

Note that the same result could be obtained by "broadening" the potentials inside the detector, by means of the substitution $V(\mathbf{r}-\mathbf{r}_n)=V_0\delta\Omega\rho(\mathbf{r})/N$, $\forall n=1,\ldots,N$, in Eq. (2.4). For the physical meaning of the smearing hypothesis refer also to Fig. 1, in which the one-dimensional model is sketched. In this way, the detector D has been transformed into a "fluid" of density $\rho(\mathbf{r})$, in which up to

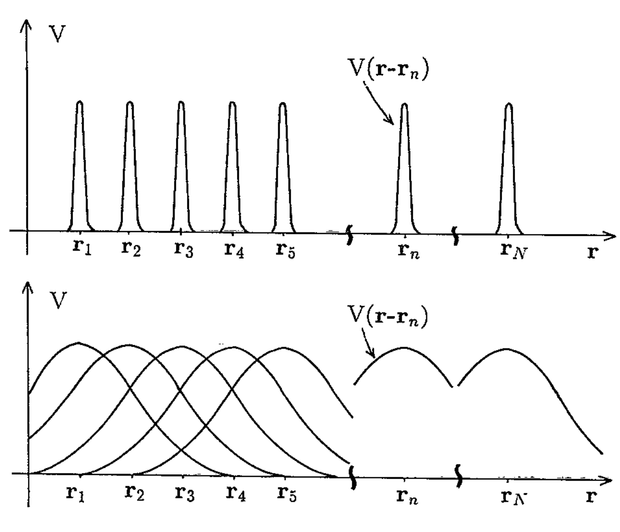

FIG. 1. "Broadening" one-dimensional potentials.

N excitations can be created.

If now we were to substitute $\rho(\mathbf{r})$ with a constant, "average" value, the original Hamiltonian H would lose any dependence on the molecules' locations. In doing so, we would overestimate the quantum-mechanical coherence of the detector, because we would neglect the additional phase randomization actually introduced by the neglected spatial degrees of freedom [9]. In order to better take into account the random motion of the elementary constituents of D, we follow a procedure applied in Ref. [10], and substitute the operator ρ with a c number plus a fluctuation

$$\rho(\mathbf{r}) \to \langle \rho \rangle + \delta\rho(\mathbf{r}) , \tag{2.9}$$

where $\langle \rho \rangle$ is a constant background density, and $\delta\rho$ obeys the following statistical properties:

$$\langle \delta\rho(\mathbf{r}) \rangle = 0, \quad \langle [\delta\rho(\mathbf{r})]^2 \rangle = F_\theta , \tag{2.10}$$

where the brackets denote an ensemble average over all the possible microscopic configurations of the detector, and θ is a critical parameter, for instance the detector temperature. The meaning of Eq. (2.9) in the present context will be given in Sec. V, where an interpretation of the wave-function collapse will be put forward. The precise mathematical expression for F_θ depends on the physical and chemical properties of D. Suffice it to know here, that the behavior of F_θ becomes critical when a phase transition takes place. Our Hamiltonian is now

$$\begin{aligned} H \simeq H' &= \tfrac{1}{2}(1+\tau_3)\frac{V_0 \delta\Omega}{N}(\langle \rho \rangle + \delta\rho) \sum_{n=1}^{N} \sigma_1^{(n)} \\ &= \tfrac{1}{2}(1+\tau_3) g \sum_{n=1}^{N} \sigma_1^{(n)} , \end{aligned} \tag{2.11}$$

where we have defined the coupling constant

$$g \equiv \frac{V_0 \delta\Omega}{N}(\langle \rho \rangle + \delta\rho) = g_0 + \delta g , \tag{2.12}$$

and the index 0 will label henceforth ideal, fluctuation-independent quantities. Notice that the coupling constant contains a small $\mathbf{r}$-dependent random part δg: The aim of this paper is just to show that its effect is far from being negligible. Before doing so, we shall summarize Cini's results.

III. REDERIVATION OF CINI'S RESULTS

The Hamiltonian (2.11) was shown [5] to be identical to the one studied by Cini [3], if we restrict our attention to the S_N-invariant sector of the Hilbert space $\mathcal{H}$ of the N molecules, where S_N is the group of permutations on $\{1,..,N\}$. It is worth remarking that this matches the analysis yielding Eq. (2.8); because every molecule is smeared out in the detector, there is no reason to consider states of $\mathcal{H}$ which are not symmetric under exchange of molecules.

Let us now rederive the main results of Cini's analysis by making use of the generalized-coherent-state approach put forward by Kudaka, Matsumoto, and Kakazu [8]. We warn the reader that the analysis in this section is only valid when g is a constant, independent of time. If the contribution of δg, in Eq. (2.12), becomes important [in the sense that after taking the ensemble average, defined in Eq. (2.10), the effect of F_θ is non-negligible] the analysis requires care and must be modified. This modification will be accomplished in the next section.

Define

$$\Sigma_j^{(N)} = \frac{1}{N} \sum_{n=1}^{N} \sigma_j^{(n)}, \quad j=1,2,3 \tag{3.1}$$

so that

$$[N\Sigma_i^{(N)}, N\Sigma_l^{(N)}] = 2iN\Sigma_k^{(N)} , \tag{3.2}$$

with i,l,k any even permutation of 1,2,3. The operators $N\Sigma_j$ are therefore a (unitary) representation of SU(2) in the Hilbert space $\mathcal{H}$. Moreover, by defining

$$\Sigma_\pm^{(N)} = \tfrac{1}{2}(\Sigma_1^{(N)} \pm i\Sigma_2^{(N)}) , \tag{3.3}$$

one gets the closed algebra [11]

$$\begin{aligned} &[N\Sigma_-^{(N)}, N\Sigma_+^{(N)}] = -N\Sigma_3^{(N)} , \\ &[N\Sigma_-^{(N)}, -N\Sigma_3^{(N)}] = -2N\Sigma_-^{(N)} , \\ &[N\Sigma_+^{(N)}, -N\Sigma_3^{(N)}] = +2N\Sigma_+^{(N)} . \end{aligned} \tag{3.4}$$

Let $|n, N-n\rangle$ be the detector state in which n molecules are in the ground state (*up*), and $N-n$ in the dissociated state (*down*). (Henceforth, we will use the terms "excited" and "dissociated" interchangeably to indicate the molecule in its *down* state.) It is then easy to compute

$$\begin{aligned} &N\Sigma_+|n,N-n\rangle = \sqrt{(n+1)(N-n)}\,|N+1, N-n-1\rangle , \\ &N\Sigma_-|n,N-n\rangle = \sqrt{n(N-n+1)}\,|n-1, N-n+1\rangle , \\ &N\Sigma_3|n,N-n\rangle = (2n-N)|n,N-n\rangle . \end{aligned} \tag{3.5}$$

The total system $Q+D$ is assumed initially to be in the state

$$\psi(0) = \chi \otimes |N,0\rangle , \tag{3.6}$$

and its evolution yields a generalized coherent state, which is readily calculated by Eqs. (3.4) and (3.5) as

$$\begin{aligned} \psi(t) &= e^{-iH't/\hbar}\psi(0) \\ &= e^{-i(1/2)(1+\tau_3)\alpha N\Sigma_1^{(N)}}\psi(0) \\ &= c_+ u_+ \phi_+(\mathbf{r}) e^{-i(\tan\alpha)N\Sigma_-^{(N)}} e^{\ln(\cos\alpha)N\Sigma_3^{(N)}} e^{-i(\tan\alpha)N\Sigma_+^{(N)}} |N,0\rangle + c_- u_- \phi_-(\mathbf{r})|N,0\rangle \\ &= c_+ u_+ \phi_+(\mathbf{r}) \sum_{n=0}^{N} a_n(t)|n,N-n\rangle + c_- u_- \phi_-(\mathbf{r})|N,0\rangle , \end{aligned}$$

$$a_n(t) = \frac{\sqrt{N!}(-i)^{N-n}}{\sqrt{n!}\sqrt{(N-n)!}} \cos^n\frac{gt}{\hbar} \sin^{N-n}\frac{gt}{\hbar} , \tag{3.7}$$

where $\alpha=gt/\hbar$, g is given by Eq (2.12), and we have used the relation [11]

$$e^{-i\alpha N\Sigma_1^{(N)}}=e^{\tanh(-i\alpha)N\Sigma_-^{(N)}}e^{\ln\cosh(-i\alpha)N\Sigma_3^{(N)}}e^{\tanh(-i\alpha)N\Sigma_+^{(N)}}$$
$$=e^{-i(\tan\alpha)N\Sigma_-^{(N)}}e^{\ln(\cos\alpha)N\Sigma_3^{(N)}}e^{-i(\tan\alpha)N\Sigma_+^{(N)}}. \quad (3.8)$$

The probability of finding n particles in the ground state at time t is given by

$$P_n(t)=|a_n(t)|^2=\begin{bmatrix}N\\n\end{bmatrix}p(t)^n q(t)^{N-n}, \quad (3.9)$$

where

$$p(t)=\cos^2\alpha(t),$$
$$q(t)=1-p(t)=\sin^2\alpha(t), \quad \alpha(t)=\frac{gt}{\hbar}. \quad (3.10)$$

For N large, the distribution of P_n is very strongly peaked around its average value

$$n_{\text{av}}(t)=\sum_{n=0}^{N}nP_n(t)\simeq Np(t), \quad (3.11)$$

and application of Stirling's formula to Eq. (3.9) gives

$$P_{n_{\text{av}}}\simeq 1. \quad (3.12)$$

Since, however, $\sum_{n=0}^{N}P_n=1$, the probability of finding $n\neq n_{\text{av}}(t)$ is negligible, and one can approximate Eq. (3.7) with

$$\psi(t)\simeq c_+u_+\phi_+(\mathbf{r})|n_{\text{av}}(t),N-n_{\text{av}}(t)\rangle$$
$$+c_-u_-\phi_-(\mathbf{r})|N,0\rangle. \quad (3.13)$$

Note that, from Eqs. (3.10) and (3.11), one gets

$$P_n(0)=\delta_{nN},\quad P_n(t_0)=\delta_{n0}, \quad (3.14)$$

where

$$t_0=\frac{\pi\hbar}{2g} \quad (3.15)$$

is the time for complete excitation of D. Finally, let us stress that application of the de Moivre–Lapace theorem to Eq. (3.9) yields

$$P_{n_{\text{av}}+\Delta n}=\frac{1}{\sqrt{2\pi Npq}}e^{-(\Delta n)^2/2Npq}, \quad (3.16)$$

for the probability of finding $n_{\text{av}}+\Delta n$ particles in the ground state, at any given time t. Equations (3.7)–(3.16) were derived in Ref. [3] in terms of a different Hamiltonian, by making use of alternative techniques. We will not pursue the analysis any further, for what has been summarized is sufficient for our purposes.

The previous results are very interesting, but suffer from a drawback: From Eq. (3.11), one obtains

$$P_n(2t_0)=\delta_{nN}, \quad (3.17)$$

so that the system eventually returns to its initial state. This is due to the Hermiticity of the Hamiltonian H', which describes a *reversible* interaction. Cini solves this problem by invoking the second law of thermodynamics. According to him, "no counter will ever recharge itself because [at $t=t_0$] all the ... ions will spontaneously recombine to form the initial neutral state." Alternatively, one might argue that at constant $\langle\rho\rangle$, by Eq. (2.12), one has $g\propto N^{-1}$ so that the time $t_0=\pi\hbar/2g$ for complete discharge turns out to be proportional to N, i.e., practically ∞, in the thermodynamical limit. In practice, a *macroscopic* detector does not go back to its initial state. In the following, we shall put these ideas into a more precise mathematical scheme. In doing so, we will argue that the decoherence of the $Q+D$ system is *not* simply due to the discharge of the counter, but has instead a statistical origin ascribable to the MHS structure of D.

IV. APPROACHING THE PHASE TRANSITION

A. Density matrix

The density matrix of the $Q+D$ system is written as

$$\Xi=|\psi\rangle\langle\psi|=\Xi_{\text{diag}}+\Xi_{\text{off}}, \quad (4.1)$$

where

$$\Xi_{\text{diag}}=|c_+|^2\eta_{++}\sum_{n=0}^{N}P_n|n,N-n\rangle\langle n,N-n|$$
$$+|c_-|^2\eta_{--}|N,0\rangle\langle N,0|,$$
$$\Xi_{\text{off}}=|c_+|^2\eta_{++}\sum_{n\neq m}a_na_m^*|n,N-n\rangle\langle m,N-m|$$
$$+c_+c_-^*\eta_{+-}\sum_{n=0}^{N}a_n|n,N-n\rangle\langle N,0|+\text{c.c.} \quad (4.2)$$

are the diagonal and off-diagonal terms, respectively, $\eta_{\pm\pm}=u_\pm u_\pm^\dagger|\phi_\pm|^2$ and $\eta_{+-}=u_+u_-^\dagger\phi_+\phi_-^*$. The coefficients a_n and the probabilities P_n are written explicitly as in Eqs. (3.7) and (3.9), respectively. Observe that there are two types of off-diagonal terms, quadratic ($a_na_m^*$) and linear (a_n). The former reflect only the coherence among different states of the detector [they would be present even if the initial state of the incoming particle had been $c_+u_+\phi_+$, instead of χ, in Eq. (2.2)], while the latter reflect the coherence between different states of the particle as well.

The analysis of the preceding section is valid as long as the contribution of δg is negligible. When the effect of F_θ, defined in Eq. (2.10), becomes important, the equations must be modified accordingly. Obviously, this effect will come into play when $t\to t_0$ [the time for complete excitation given in Eq. (3.15)], because in that case the detector approaches a highly unstable state in which (almost) all of its elementary constituents are in their excited state, and the behavior of F_θ becomes critical. Our problem is to understand what happens to the coefficient $a_n(t)$, in Eq. (3.7), when the detector approaches a phase transition. Throughout this section, the ensemble average $\langle\ \rangle$ of Eq. (2.10) will be calculated for the relevant physical quantities. Its physical meaning in the present

present context will be given in the next section, where we will argue that it corresponds to an average over many repetitions of the same experiment.

In conformity with Eqs. (2.12) and (3.10), define

$$\alpha(t)=(g_0+\delta g)\frac{t}{\hbar}=\alpha_0(t)+\delta\alpha(t)\ , \tag{4.3}$$

where, from Eqs. (2.12) and (2.10),

$$\langle(\delta\alpha)^2\rangle=\left[\frac{V_0\delta\Omega t}{N\hbar}\right]^2 F_\theta\equiv f\ . \tag{4.4}$$

In the following we shall assume the validity of the Gaussian approximation

$$\langle(\delta\alpha)^n\rangle=\begin{cases}0 & \text{for odd } n\\ \langle(\delta\alpha)^2\rangle^k(2k)!/2^k k! & \text{for even } n=2k\ .\end{cases} \tag{4.5}$$

B. Diagonal elements

Let us start from a preliminary observation. From Eq. (A3), derived in the Appendix, we get

$$I_{2n}^{2N}\equiv\langle\cos^{2n}(\alpha_0+\delta\alpha)\sin^{2N-2n}(\alpha_0+\delta\alpha)\rangle$$
$$=e^{(1/2)fD_\alpha^2}\cos^{2n}\alpha_0\sin^{2N-2n}\alpha_0\ , \tag{4.6}$$

so that, for $N=n=1$, we obtain by Eq. (3.11),

$$\langle n_{\rm av}(t)\rangle=N\langle\cos^2(\alpha_0+\delta\alpha)\rangle$$
$$=(N/2)(1+e^{-2f}\cos2\alpha_0)\xrightarrow[f\to\infty]{}N/2\ . \tag{4.7}$$

This is rather curious. The ensemble average ($\langle\ \rangle$) of the maxima of the distributions ($n_{\rm av}$) tends to $N/2$, for $t\to t_0$, and *not* to 0, as one might suppose [12]. In order to understand the meaning of this result, we must pursue our analysis a little further.

From Eq. (A8) in the Appendix, we know that

$$\langle P_n(t)\rangle=\langle|a_n(t)|^2\rangle$$
$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx\,e^{-x^2}\begin{bmatrix}N\\ n\end{bmatrix}\cos^{2n}(\sqrt{2f}x+\alpha_0)\sin^{2N-2n}(\sqrt{2f}x+\alpha_0)\ , \tag{4.8}$$

and in the large-N limit we get, in the Stirling approximation,

$$\langle P_n\rangle\simeq\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx\,e^{-x^2}\delta(n-N\cos^2(\sqrt{2f}x+\alpha_0))$$
$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx\,e^{-x^2}\frac{1}{|2N\sin(\sqrt{2f}x+\alpha_0)\cos(\sqrt{2f}x+\alpha_0)|_{x=x_0}\sqrt{2f}}\sum_{x_0}\delta(x-x_0)\ , \tag{4.9}$$

where x_0 denotes the solutions of the equation $\cos(\sqrt{2f}x_0+\alpha_0)=\pm\sqrt{n/N}$. An easy calculation yields

$$\langle P_n\rangle\simeq\frac{1}{2\sqrt{\pi}\sqrt{n(N-n)}\sqrt{2f}}\times\int_{-\infty}^{\infty}dx\,e^{-x^2}\sum_{n=-\infty}^{\infty}\delta\left[x-\frac{1}{\sqrt{2f}}(n\pi\pm\alpha^*-\alpha_0)\right], \tag{4.10}$$

where $\alpha^*=\arccos\sqrt{n/N}$ ($0\le\alpha^*\le\pi/2$). The integral in Eq. (4.10) can be evaluated by observing that, due to the exponential term, the main contribution comes from the interval $-2\lesssim x\lesssim 2$. When $t\ll t_0$, f is very small, and therefore in Eq. (4.10) only a few δ functions (at most) have a vanishing argument within the above-mentioned interval. On the other hand, for $t\to t_0$, we have $f\to\infty$, so that more and more zeros lie within the relevant interval $-2\lesssim x\lesssim 2$. In particular, it is easy to verify that if f is sensibly different from 0, the number Δn of zeros lying within the interval is $\Delta n\simeq 4\sqrt{2f}/\pi$, so that

$$\langle P_n\rangle\simeq\frac{1}{\pi\sqrt{n(N-n)}}\ , \tag{4.11}$$

which is the distribution sought. Here the normalization condition $\int_0^N\langle P_n\rangle dn=1$ has been imposed again. A few remarks are now in order. Let us start by observing that the divergencies at $n=0,N$ do not worry us. Indeed, for discrete n, the probability should be defined as

$$\langle\mathcal{P}_n\rangle\equiv\int_n^{n+1}\langle P_m\rangle dm$$
$$=\frac{2}{\pi}\left[\arccos\left[\frac{n}{N}\right]^{1/2}-\arccos\left[\frac{n+1}{N}\right]^{1/2}\right], \tag{4.12}$$

so that

$$\langle\mathcal{P}_0\rangle=\langle\mathcal{P}_{N-1}\rangle\simeq(2/\pi)N^{-1/2}\ . \tag{4.13}$$

Moreover,

$$\langle n\rangle_{\rm av}=\int_0^N n\langle P_n\rangle dn\simeq N/2\ , \tag{4.14}$$

in agreement with our previous result [Eq. (4.7)]. Comparison of Eqs. (4.7) and (4.14) allows us to infer that the two averages $\langle\ \rangle$ and $(\ )_{\rm av}$ can be computed in either order, if care is taken in calculating the relevant limits: The ensemble average $\langle\ \rangle$ of the mean values $(\ )_{\rm av}$ [Eq. (4.7)] is equal, in the $f\to\infty$ limit, to the mean value of the ensemble averages [Eq. (4.14)]. It is worth stressing that only the ensemble average $\langle\ \rangle$ has relevance for the loss of quantum-mechanical coherence, while the average $(\ )_{\rm av}$

is essentially the same as the one introduced in Eq. (3.11).

It is also interesting to remark that the final distribution is almost flat. Indeed

$$\langle \mathcal{P}_{N/2}\rangle \simeq (2/\pi)N^{-1} , \tag{4.15}$$

which should be compared to Eq. (4.13), and the standard deviation is given by

$$\sigma = \left[\int_0^N (n-\langle n\rangle_{\rm av})^2 \langle P_n\rangle dn \right]^{1/2} = 2^{-3/2}N . \tag{4.16}$$

This result is somewhat unexpected, and could not be guessed on the basis of the previous analyses [3,6–8]. It is a consequence of the approximation $H' \gg H_Q, H_D$, in Eqs. (2.3) and (2.4). To neglect H_D means to consider the limit $\hbar\omega \to 0$, i.e., the limit of vanishing energy difference between the excited and ground states of the detector's elementary constituents. Had such a limit not been taken, one should have expected the detector's state to evolve towards the minimum energy state, i.e., the initial state $|N,0\rangle$. But in our case *all the detector's states are energetically equivalent,* and the resulting distribution is almost flat.

C. Quadratic off-diagonal elements

Our next task is now to evaluate $\Xi_{\rm off}$, the off-diagonal part of the density matrix Ξ. Consider, for $m \neq n$

$$\begin{aligned} \langle a_n a_m^* \rangle &= i^{n-m} \begin{bmatrix} N \\ n \end{bmatrix}^{1/2} \begin{bmatrix} N \\ m \end{bmatrix}^{1/2} \langle \cos^{n+m}\alpha \sin^{2N-(n+m)}\alpha \rangle \\ &= i^{n-m} \begin{bmatrix} N \\ n \end{bmatrix}^{1/2} \begin{bmatrix} N \\ m \end{bmatrix}^{1/2} \\ &\quad \times \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \cos^{n+m}\alpha' \sin^{2N-(n+m)}\alpha' , \end{aligned} \tag{4.17}$$

where $\alpha' = \sqrt{2f}\,x + \alpha_0$, and we have used again Eq. (A8). The modulus of this quantity is evaluated as

$$\begin{aligned} |\langle a_n a_m^* \rangle| \le & \begin{bmatrix} N \\ n \end{bmatrix}^{1/2} \begin{bmatrix} N \\ m \end{bmatrix}^{1/2} \\ & \times \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} |\cos^n\alpha'| |\sin^{N-n}\alpha'| \\ & \times |\cos^m\alpha'| |\sin^{N-m}\alpha'| . \end{aligned} \tag{4.18}$$

We know that, for N large, the binomial distribution with $p = \cos^2\alpha'$, $q = \sin^2\alpha'$ [Eq. (3.9)] tends to the Gaussian distribution of Eq. (3.16). Consequently

$$\begin{aligned} \left[\begin{bmatrix} N \\ k \end{bmatrix} \cos^{2k}\alpha' \sin^{2N-2k}\alpha' \right]^{1/2} &= \begin{bmatrix} N \\ k \end{bmatrix}^{1/2} |\cos\alpha'|^k |\sin\alpha'|^{N-k} \\ &\xrightarrow[\text{large N}]{} \left[\frac{e^{-(k-N\cos^2\alpha')^2/2N\cos^2\alpha'\sin^2\alpha'}}{(2\pi N \cos^2\alpha' \sin^2\alpha')^{1/2}} \right]^{1/2} . \end{aligned} \tag{4.19}$$

Substitution into Eq. (4.18) yields

$$|\langle a_n a_m^* \rangle| \lesssim \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \frac{e^{-(n-N\cos^2\alpha')^2/4N\cos^2\alpha'\sin^2\alpha'} e^{-(m-N\cos^2\alpha')^2/4N\cos^2\alpha'\sin^2\alpha'}}{(2\pi N\cos^2\alpha'\sin^2\alpha')^{1/2}} . \tag{4.20}$$

Since N is large, the two Gaussians in the integral are very strongly peaked around those values of α' such that $N\cos^2\alpha' = n, m$, respectively. Moreover, the solutions of these two equations form two disjoint sets, so that the integral vanishes for $n \neq m$. Observe that the above argument is strictly valid only for n "macroscopically different" from m. A residual (though small) coherence is still present between states with $n \simeq m$. In order to estimate it, write $m = n + \Delta n$, in Eq. (4.20). Application of Stirling's formula yields

$$\begin{aligned} |\langle a_n a_{n+\Delta n}^* \rangle| &\lesssim \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \frac{e^{-(n-N\cos^2\alpha')^2/2N\cos^2\alpha'\sin^2\alpha'}}{(2\pi N\cos^2\alpha'\sin^2\alpha')^{1/2}} e^{-[(\Delta n)^2 + 2\Delta n(n-N\cos^2\alpha')]/4N\cos^2\alpha'\sin^2\alpha'} \\ &\simeq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} \delta(n-N\cos^2\alpha') e^{-(\Delta n)^2/4N\cos^2\alpha'\sin^2\alpha'} \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2} \delta(n-N\cos^2\alpha') e^{-[N/4n(N-n)](\Delta n)^2} , \end{aligned} \tag{4.21}$$

where we have substituted $\cos^2\alpha' = n/N$ and $\sin^2\alpha' = 1 - n/N$ in the last equality. Moreover, observe that

$$\frac{1}{N} \le \frac{N}{4n(N-n)} \le \infty \quad \text{for } 0 \le n \le N , \tag{4.22}$$

where the lower limit is obtained for $n = N/2$, the upper one for $n = 0, N$. The largest values of the right-hand side in Eq. (4.21) are thus when $n \simeq m \simeq N/2$, for which one has

$$|\langle a_n a^*_{n+\Delta n}\rangle| \lesssim \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\delta(n-N\cos^2\alpha')e^{-(\Delta n)^2/N}$$
$$\simeq \frac{1}{\pi\sqrt{n(N-n)}}e^{-(\Delta n)^2/N}$$
$$\simeq \frac{2}{\pi N}e^{-(\Delta n)^2/N} \quad (n\simeq N/2)\,, \tag{4.23}$$

where we have used Eqs. (4.9)–(4.11). This is the upper bound sought. Summarizing:

$$|\langle a_n \alpha^*_m\rangle| \lesssim \frac{e^{-[N/4n(N-n)](n-m)^2}}{\pi\sqrt{n(N-n)}} \simeq \begin{cases} 0 & \text{for } |n-m|\gg 1 \\ 0 & \text{for } n\simeq m\simeq 0,N \\ \text{const}\times N^{-1}e^{-(n-m)^2/N} & \text{for } n\simeq m\simeq N/2\,. \end{cases} \tag{4.24}$$

Notice that for $n=m$ one recovers Eq. (4.11).

D. Linear off-diagonal elements

Finally, let us consider the coefficients a_n. By Eq. (A8), we have

$$\langle a_n\rangle = (-i)^{N-n}\begin{bmatrix} N \\ n\end{bmatrix}^{1/2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\cos^n\alpha'\sin^{N+n}\alpha'\,, \tag{4.25}$$

where $\alpha' = \sqrt{2f}x+\alpha_0$. For N large,

$$|\langle a_n\rangle| \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\left[\begin{bmatrix} N \\ n\end{bmatrix}\cos^{2n}\alpha'\sin^{2N-2n}\alpha'\right]^{1/2}$$
$$\simeq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\left[\frac{e^{-(n-N\cos^2\alpha')^2/2N\cos^2\alpha'\sin^2\alpha'}}{(2\pi N\cos^2\alpha'\sin^2\alpha')^{1/2}}\right]^{1/2}$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\frac{e^{-(n-N\cos^2\alpha')^2/4N\cos^2\alpha'\sin^2\alpha'}}{(2\pi N\cos^2\alpha'\sin^2\alpha')^{1/4}}\frac{(4\pi N\cos^2\alpha'\sin^2\alpha')^{1/2}}{(4\pi N\cos^2\alpha'\sin^2\alpha')^{1/2}}$$
$$\simeq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\delta(n-N\cos^2\alpha')(8\pi N\cos^2\alpha'\sin^2\alpha')^{1/4}$$
$$\simeq \left[\frac{8\pi n(N-n)}{N}\right]^{1/4}\frac{1}{\pi\sqrt{n(N-n)}}\,, \tag{4.26}$$

where we have used again Eqs. (4.9)–(4.11). In conclusion

$$|\langle a_n\rangle| \lesssim \frac{1}{\pi}\left[\frac{8\pi}{Nn(N-n)}\right]^{1/4}. \tag{4.27}$$

E. Loss of coherence

Let us summarize what has been achieved so far. The behavior of the density matrix of the $Q+D$ system is given by

$$\Xi \to \Xi^{(f)} = |c_+|^2\eta_{++}\sum_{n,m=0}^{N}\langle a_n a^*_m\rangle|n,N-n\rangle\langle m,N-m| + |c_-|^2\eta_{--}|N,0\rangle\langle N,0|$$
$$+c_+c^*_-\eta_{+-}\sum_{n=0}^{N}\langle a_n\rangle|n,N-n\rangle\langle N,0| + \text{H.c.}$$
$$=\Xi^{(f)}_{\text{diag}}+\Xi^{(f)}_{\text{off}}\,, \tag{4.28}$$

with

$$\Xi^{(f)}_{\text{diag}} = |c_+|^2\eta_{++}\sum_{n=0}^{N}\langle|a_n|^2\rangle|n,N-n\rangle\langle n,N-n| + |c_-|^2\eta_{--}|N,0\rangle\langle N,0|\,,$$
$$\Xi^{(f)}_{\text{off}} \simeq |c_+|^2\eta_{++}\sum_{\substack{n,m\simeq N/2 \\ n\ne m}}\langle a_n a^*_m\rangle|n,N-n\rangle\langle m,N-m| + c_+c^*_-\eta_{+-}\sum_{n=0}^{N}\langle a_n\rangle|n,N-n\rangle\langle N,0| + \text{H.c.}\,, \tag{4.29}$$

where $\langle |a_n|^2 \rangle$, $|\langle a_n a_m^* \rangle|$, and $|\langle a_n \rangle|$ are upper bounded as in Eqs. (4.11), (4.24), and (4.27), respectively. The above result is obtained in the limit of large N, when F_θ in Eq. (2.10) becomes sensibly different from 0 [see the short discussion after Eq. (4.10)]. Observe that Eqs. (2.9) and (2.10) make sense only in the large-N limit.

We can see that both diagonal and off-diagonal terms of the density matrix tend to 0 in the $N \to \infty$ limit, and of course

$$\mathrm{Tr}_{\mathrm{detector}} \Xi^{(f)} = |c_+|^2 \eta_{++} + |c_-|^2 \eta_{--} , \tag{4.30}$$

independently of N. It is important to stress that Eqs. (4.28) and (4.29) have been obtained *without* taking any partial trace over the detector states; however, it is *not* possible to state that

$$\Xi \underset{N\to\infty}{\longrightarrow} \Xi^{(f)}_{\mathrm{diag}} , \tag{4.31}$$

because $\Xi^{(f)}_{\mathrm{off}}$ is non-negligible compared to $\Xi^{(f)}_{\mathrm{diag}}$, even in the $N \to \infty$ limit. It is then apparent that comparison among coefficients is not sufficient to draw general conclusions about the behavior of the density matrix in the $N \to \infty$ limit. Let us therefore estimate the value of

$$\mathrm{Tr}(\Xi^{(f)})^2 = |c_+|^4 \sum_{n,m=0}^{N} |\langle a_n a_m^* \rangle|^2 + 2|c_+ c_-^*|^2 \sum_{n=0}^{N} |\langle a_n \rangle|^2 + |c_-|^4 , \tag{4.32}$$

where the trace is taken over both D and Q. If this is found to be less than 1, we can conclude that the density matrix is (partially) mixed, and coherence has been lost to an extent estimated by the contribution of the off-diagonal matrix elements to the value of the trace. Let us first observe that if no average $\langle\ \rangle$ were present, one would get

$$\begin{aligned}\mathrm{Tr}(\Xi^{(f)})^2 &= |c_+|^4 \sum_{n,m=0}^{N} |a_n|^2 |a_m|^2 + 2|c_+ c_-^*|^2 \sum_{n=0}^{N} |a_n|^2 + |c_-|^4 \\ &= |c_+|^4 + 2|c_+ c_-^*|^2 + |c_-|^4 \\ &= (|c_+|^2 + |c_-|^2)^2 = 1,\end{aligned} \tag{4.33}$$

as expected for a pure state. On the other hand, in our case, from Eq. (4.24) one gets

$$\begin{aligned}\sum_{n,m=0}^{N} |\langle a_n a_m^* \rangle|^2 &\lesssim \sum_{n,m=0}^{N} \left[\frac{e^{-[N/4n(N-n)](n-m)^2}}{\pi\sqrt{n(N-n)}} \right]^2 \\ &= \sum_{n,m=0}^{N} \frac{e^{-[N/2n(N-n)](n-m)^2}}{\pi^2 n(N-n)} ,\end{aligned} \tag{4.34}$$

which can be approximated as an integral over $x \equiv n/N$ and $y \equiv m/N$,

$$\begin{aligned}\sum_{n,m=0}^{N} |\langle a_n a_m^* \rangle|^2 &\lesssim N^2 \int_0^1 dx \int_0^1 dy \frac{e^{-[N/2x(1-x)](x-y)^2}}{\pi^2 N^2 x(1-x)} \\ &< \frac{1}{\pi^2} \int_0^1 dx \frac{1}{x(1-x)} \int_{-\infty}^{\infty} dy\, e^{-[N/2x(1-x)]y^2} \\ &= \frac{1}{\pi^2} \left[\frac{2\pi}{N} \right]^{1/2} \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \\ &= \left[\frac{2}{\pi N} \right]^{1/2} \underset{N\to\infty}{\longrightarrow} 0 .\end{aligned} \tag{4.35}$$

Notice that we only need to estimate the off-diagonal contribution in the above to see how the coherence is lost. We need not worry about the possible divergences appearing in the integration when $x = y = 0$ and 1, which rather belong to the diagonal contribution. To evaluate the diagonal contribution properly, we should go back to the definition of $\langle \mathcal{P}_n \rangle$ [Eq. (4.12)] and calculate $\sum_{n=0}^{N-1} \langle \mathcal{P}_n \rangle^2$. It is not difficult to show that this quantity is indeed finite and behaves like $1/N$ in the $N \to \infty$ limit. Moreover, from Eq. (4.27),

$$\begin{aligned}\sum_{n=0}^{N} |\langle a_n \rangle|^2 &\lesssim \sum_{n=0}^{N} \left[\frac{1}{\pi} (8\pi)^{1/4} \left[\frac{1}{Nn(N-n)} \right]^{1/4} \right]^2 \\ &\simeq N \int_0^1 dx \frac{\sqrt{8\pi}}{\pi^2} \left[\frac{1}{N^3 x(1-x)} \right]^{1/2} \\ &= \left[\frac{8}{\pi N} \right]^{1/2} \underset{N\to\infty}{\longrightarrow} 0 .\end{aligned} \tag{4.36}$$

These upper bounds Eqs. (4.35) and (4.36) clearly demonstrate that the coherence is lost like $1/\sqrt{N}$ when N becomes large. From Eq. (4.32), we finally obtain

$$\begin{aligned}\mathrm{Tr}(\Xi^{(f)})^2 &\lesssim |c_+|^4 \left[\frac{2}{\pi N} \right]^{1/2} + 2|c_+ c_-^*|^2 \left[\frac{8}{\pi N} \right]^{1/2} + |c_-|^4 \\ &\underset{N\to\infty}{\longrightarrow} |c_-|^4 < |c_-|^2 < 1 ,\end{aligned} \tag{4.37}$$

as was to be expected for a completely mixed state. In the model here proposed, and in the sense given by Eq. (4.37), we can state that in the $N \to \infty$ limit, coherence is lost among different detector states as well as between the two branch waves of the Q particle.

V. MANY HILBERT SPACES

At this stage, some questions are still unanswered. What is the meaning of the average $\langle\ \rangle$, introduced in Eq. (2.10)? How is coherence lost between the interfering

ing terms in Eq. (3.7)? In what sense is this coherence lost? We shall try to answer these questions by making use of the MHS approach put forward by Machida and Namiki about a decade ago [4]. Let us briefly sketch the most relevant points of their approach in the present context by following the line of thought of Ref. [13].

Consider again the two-level quantum system Q, introduced in Eq. (2.2):

$$\chi=c_+u_+\phi_+(\mathbf{r})+c_-u_-\phi_-(\mathbf{r}) . \tag{5.1}$$

The system Q has already undergone a spectral decomposition, and it is important to realize that at this stage the quantum-mechanical coherence is *fully kept,* because in principle the two branch waves can be recombined, yielding again the state χ_0 of Eq. (2.1). This point has often been a source of misunderstanding in the past, as emphasized in Ref. [7]. Let Q interact with the detector D, and

$$\psi(t)=c_+u_+\phi_+(\mathbf{r})\sum_{n=0}^{N}a_n(g,t)|n,N-n\rangle + c_-u_-\phi_-(\mathbf{r})|N,0\rangle \tag{5.2}$$

be the final state of the system $Q+D$, where the coefficients $a_n(g,t)$ depend on $g=g_0+\delta g$ as in Eq. (3.7). As discussed in the previous sections, Q and D interact for a certain period of time t_0, at the end of which D approaches a highly excited state. For the sake of simplicity, let us refer henceforth to Q as the "incoming particle." We wish to stress that this terminology is inappropriate, in the present context, because *only one* of the two branch waves describing Q undergoes an interaction with D. The reader should bear this in mind during the following discussion.

Equation (5.2) holds for *every single* quantum system Q, i.e., for every different incoming particle; let us label the different incoming particles with j ($j=1,\ldots,N_p$, where N_p is the total number of particles in an experimental run). Quantum mechanics gives information about the result of many repetitions of the *same* experiment. But the meaning of the words "*same* experiment" is unclear, when macroscopic apparatuses are involved. For instance, it is *practically impossible* to put a macroscopic apparatus in the *same* microscopic state every time the experiment is reperformed. In the example proposed in this paper, the elementary constituents of the detector D undergo internal motions that cannot be kept under control. Every time we repeat the experiment the density $\rho(\mathbf{r})$, defined in Eq. (2.6), will assume slightly different values, which reflect the different microscopic configurations of the detector corresponding to the "same" (macroscopic) state. In this sense, the density ρ is a function of j, the index labeling different incoming particles. Accordingly, Eq. (2.9) should be rewritten as

$$\rho^{(j)}(\mathbf{r})\to\langle\rho\rangle+\delta\rho^{(j)}(\mathbf{r}),\quad j=1,\ldots,N_p , \tag{5.3}$$

because the constant background density $\langle\rho\rangle$ does not change when the "same" experiment is reperformed, but the fluctuating part $\delta\rho$ will assume different values for different incoming particles. Consequently, Eq. (5.2) reads

$$\psi^{(j)}(t)=c_+u_+\phi_+(\mathbf{r})\sum_{n=0}^{N}a_n^{(j)}(g,t)|n,N-n\rangle + c_-u_-\phi_-(\mathbf{r})|N,0\rangle,\quad j=1,\ldots,N_p , \tag{5.4}$$

where the j dependence of the coefficients a_n has been written explicitly. On the other hand, as already stressed, quantum mechanics gives information only for many repetitions of the same experiment. (In this sense it is a statistical theory in that, except in the very special cases in which the system is in an eigenstate of the observable to be measured, nothing can be said about a single event.) This leads us to consider the average over j as the *only experimentally meaningful quantity:* For every j-dependent quantity b, define the average over many different incoming particles as

$$\overline{b}\equiv\frac{1}{N_p}\sum_{j=1}^{N_p}b_j . \tag{5.5}$$

The ergodic assumption

$$\overline{\cdots}=\langle\cdots\rangle , \tag{5.6}$$

where $\langle\cdots\rangle$ is the ensemble average defined in Eq. (2.10), provides us with a consistent interpretation of the results in Sec. IV: When the same experiment is performed many times on many $Q+D$ systems, the quantum-mechanical coherence is lost at a *statistical level,* during the interaction. In particular, the coefficients $a_n^{(j)}(g,t)$, for different j's, will undergo random fluctuations, and on the average we obtain

$$\begin{aligned}\overline{|a_n(g,t)|^2}&=\frac{1}{N_p}\sum_{j=1}^{N_p}|a_n^{(j)}(g,t)|^2\\&=\langle|a_n(g,t)|^2\rangle,\quad\text{i.e., }\overline{P_n}=\langle P_n\rangle ,\\\overline{a_na_m^*}&=\langle a_na_m^*\rangle ,\\\overline{a_n}&=\langle a_n\rangle .\end{aligned} \tag{5.7}$$

The loss of coherence, here represented by a flat distribution of the detector's final states [Eqs. (4.11) and (4.16)], and by the behavior of $\mathrm{Tr}(\Xi^{(f)})^2$ in the $N\to\infty$ limit [Eq. (4.37)] is just a statistical effect over many repetitions of the experiment. It is worth stressing that the results in Sec. IV hold true for any nonzero values of f and F_θ: We shall come back to this important point in the final discussion.

Notice that we are not discussing the limits of validity of the ergodic assumption, in Eq. (5.6). In particular it should be emphasized that Eq. (5.5) makes sense only if N_p, the total number of particles in an experimental run, is very large. Stated differently, it makes no sense to speak of wave-function collapse for a single particle, because here the very concept of wave-function collapse is replaced by the idea of a *statistically defined* dephasing process over a large ensemble of particles. For a discussion on this point, see Ref. [13].

VI. CONCLUDING REMARKS

We have shown how the interaction between a quantum system Q and a detector D can provoke a loss of coherence. In the example proposed, the interaction Hamiltonian is such that only the "up" state of Q interacts with D, and this interaction provokes a phase transition in D. The interaction is such that, *on the average,* the final state of the detector can be any of the possible states $|n,N-n\rangle$, with a probability $\langle P_n\rangle$ given by Eq. (4.11).

At this point, one might wonder about the validity of our analysis. Indeed, under our approximations, the Hamiltonian in Eq (2.11) is given in terms of a coupling constant g containing a fluctuating term δg, as in Eq. (2.12), and it is just this fluctuating part which becomes responsible for the loss of coherence. The coupling constant is therefore not really a "constant" because, by definition, it reflects the internal motions of the detector's elementary constituents. Strictly speaking, one should take these internal motions into account *ab initio,* and solve the N-body problem directly; the set of assumptions leading to Eq. (2.11) excuses us from solving explicitly this formidably involved problem. Then to what extent is our final result sensible? The answer comes from our discussion in Sec. V. For every Q system (namely, for every j), the evolution can be computed exactly as in Eq. (5.4). Of course, the coefficients $a_n^{(j)}(g,t)$ will depend on ρ [Eq. (5.3)], and will therefore be functions of $\mathbf{r}$ and t through $\delta\rho$; but at *fixed* j the contribution of $\delta\rho$ can *always be considered small* when compared to $\langle\rho\rangle$, even for $t\rightarrow t_0$. In this sense, *at fixed* j, the coupling constant is indeed a "constant" to a very good approximation.

The loss of coherence is only an *average effect* that takes place when the ergodic hypothesis (5.6) holds, i.e., when N_p, the number of repetitions of the experiment, is large enough. Stated differently, the quantum-mechanical problem of determining the evolution of the $Q+D$ system is solved for fixed j, as it should be; it is only when this evolution has been determined, for every time t, that we can compute the ensemble average defined in Eq. (2.10). By this argument, the divergence of F_θ and f for $t\rightarrow t_0$ does not invalidate the mathematical soundness of our analysis.

It is also worth stressing, at this point, that in this approach, as well as in the MHS "philosophy," *no average should be taken before the evolution is completely determined.* This is a significant difference to many other proposals in the literature in which averages and/or partial traces are often computed at the beginning of the calculation. It is our opinion that in the latter way one runs the risk of introducing the final results (the wave-function collapse) in the very premises, instead of deriving it. We should also remark that the thermodynamical limit $N\rightarrow\infty$ has been considered only at the end of Sec. IV E in connection with the evaluation of $\mathrm{Tr}(\Xi^{(f)})^2$. A finite detector containing a finite number of elementary constituents suffices for all other purposes, within the limits of validity of Stirling's formula.

We shall put forward a final remark. Is a phase transition necessary to produce the loss of coherence? The answer is no. The phase transition, as described in this paper, corresponds to the *discharge* of the counter, not to the collapse of the wave function. The collapse, i.e., the loss of coherence, begins already at $t=0$, immediately after the interaction has been "switched on." Indeed, the coefficients $a_n(t)$ in Eq. (3.7) start experiencing the presence of noise, via the coupling constant g, as soon as the evolution governed by the Hamiltonian H' begins. The loss of coherence engendered by this process is *partial,* but *unrecoverable,* and we can say that a *partial collapse of the wave function* begins as soon as the microsystem enters into contact with the (macroscopic) detector. The final discharge of D, at $t=t_0$, is therefore a different effect that follows a process started long before. In this sense we think that a precise distinction should be made between the discharge of the counter (the "signal" or in general the "display" of the result of the experiment) and the collapse of the wave function (the "decoherence" of the quantum system, i.e., the transition from pure to mixed state).

It is also necessary to stress that the results derived in the present paper clearly call for a statistical interpretation of the wave-function collapse. The dephasing process here described is such that coherence is lost *statistically* in *repetitions* of a measurement. In this sense the "collapse" is not to be regarded as a real physical process, but rather as a modification of the density matrix describing the total system considered. Indeed, one can even argue that a mean value (an average) is definitely not a property of reality, but only a useful operational concept.

We preferred not to address these philosophical issues in our discussion: In fact, the problem of the meaning of "physical reality" is far too difficult to be tackled and resolved by our discussion. More so, our analysis is model dependent, and it is always risky to abstract the characteristics of a rather simple example in order to draw general conclusions. Nevertheless, we believe that the model here proposed can give some useful insights into the physical mechanism engendering the loss of coherence suffered by a quantum-mechanical system when interacting with a macroscopical apparatus. Moreover, it enforces the idea that it is possible to dispense with von Neumann's projection rules and with the presence of an external, "classical" observer.

Finally, we wish to emphasize that two important points have been overlooked in our analysis. First, we have assumed N fixed during the interaction between Q and D, which is clearly unphysical. Indeed, a detector is a macroscopic body, and as such it cannot be considered isolated from the external world. In a more realistic treatment the detector should be considered as an *open system,* continuously exchanging matter and energy with its "environment." Consequently, the number N of elementary constituents of D should be allowed to fluctuate around some average value. These points are thoroughly discussed in Refs. [4] and [14] in terms of superselection rules. Second, we have not attempted to estimate how large N should be in order to produce a noteworthy loss of coherence. Under the conditions just described, a macroscopic detector can be a suitable object to collapse the wave function. On the other hand, this does not mean that, say, a few thousands (or even hundreds) of molecules would not provoke a (partial) loss of coherence. Indeed Stirling's formula is asymptotic, and as such it holds to a fairly good degree of accuracy already for rather small N. The problem of understanding in what sense and to what extent coherence can be lost (*partial* collapse of the wave function) when a quantum system interacts with a relatively small number of particles is still completely open, and would require a different analysis.

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APPENDIX

In this appendix we shall derive Eqs. (4.6) and (4.8). By making use of the displacement operator, $e^{\delta\alpha D_\alpha}$, with $D_\alpha \equiv \partial/\partial\alpha_0$, we can write

$$\begin{aligned} I_m^M &\equiv \langle \cos^m(\alpha_0+\delta\alpha)\sin^{M-m}(\alpha_0+\delta\alpha)\rangle \\ &= \langle e^{\delta\alpha D_\alpha}\cos^m\alpha_0 \sin^{M-m}\alpha_0\rangle \\ &= \langle e^{\delta\alpha D_\alpha}\rangle \cos^m\alpha_0\sin^{M-m}\alpha_0 , \end{aligned} \tag{A1}$$

and by the Gaussian properties of the fluctuation $\delta\alpha$ [Eq. (4.5)] we get

$$\begin{aligned} \langle e^{\delta\alpha D_\alpha}\rangle &= \sum_{n=0}^{\infty} \frac{\langle(\delta\alpha)^n\rangle}{n!} D_\alpha^n \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{f}{2}D_\alpha^2\right]^k \\ &= e^{(1/2)fD_\alpha^2} , \end{aligned} \tag{A2}$$

where $f \equiv \langle(\delta\alpha)^2\rangle$ is defined in (4.4). Therefore we obtain

$$I_m^M = e^{(1/2)fD_\alpha^2}\cos^m\alpha_0\sin^{M-m}\alpha_0 , \tag{A3}$$

which proves Eq. (4.6). Equation (A3) can be rewritten as

$$\begin{aligned} I_m^M = \left[\frac{1}{2}\right]^m \left[\frac{1}{2i}\right]^{M-m} & e^{(1/2)fD_\alpha^2}(e^{i\alpha_0}+e^{-i\alpha_0})^m \\ & \times(e^{i\alpha_0}-e^{-i\alpha_0})^{M-m}e^{-(1/2)fD_\alpha^2} , \end{aligned} \tag{A4}$$

where the operator $e^{-(1/2)fD_\alpha^2}$ (the effect of which is trivial) has been added for later convenience. Expanding the above formula in powers of $e^{i\alpha_0}$ and using the formulas

$$\begin{aligned} & e^{(1/2)fD_\alpha^2}\alpha_0 e^{-(1/2)fD_\alpha^2} = \alpha_0 + fD_\alpha , \\ & e^{ic(\alpha_0+fD_\alpha)} = e^{ic\alpha_0}e^{-(c^2/2)f}e^{icfD_\alpha} \quad (c \text{ const}) \end{aligned} \tag{A5}$$

we obtain

$$\begin{aligned} I_m^M = \left[\frac{1}{2}\right]^m \left[\frac{1}{2i}\right]^{M-m} \sum_{r=0}^{m}\sum_{l=0}^{M-m} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} M-m \\ l \end{bmatrix} (-1)^l & \\ \times e^{-(M-2r-2l)^2(1/2)f} & \\ \times e^{i(M-2r-2l)\alpha_0} & , \end{aligned} \tag{A6}$$

where the irrelevant factor $e^{i(M-2r-2l)fD_\alpha}$ has been suppressed. Next, we linearize the exponent by introducing inside the summations over r and l the identity

$$1 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-[x-i(M-2r-2l)\sqrt{(1/2)f}\,]^2} . \tag{A7}$$

After performing the summations we finally arrive at the desired expression

$$\begin{aligned} I_m^M &= \left[\frac{1}{2}\right]^m \left[\frac{1}{2i}\right]^{M-m} \sum_{r=0}^{m}\sum_{l=0}^{M-m} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} M-m \\ l \end{bmatrix} (-1)^l \\ &\qquad \times \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}} e^{-[x-i(M-2r-2l)\sqrt{(1/2)f}\,]^2} e^{-(M-2r-2l)^2(1/2)f} e^{i(M-2r-2l)\alpha_0} \\ &= \left[\frac{1}{2}\right]^m \left[\frac{1}{2i}\right]^{M-m} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx e^{-x^2} \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} e^{-i(\sqrt{2f}x+\alpha_0)2r} \sum_{l=0}^{M-m} \begin{bmatrix} M-m \\ l \end{bmatrix} (-1)^l e^{i(\sqrt{2f}x+\alpha_0)(M-2l)} \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx e^{-x^2}\cos^m(\sqrt{2f}x+\alpha_0)\sin^{M-m}(\sqrt{2f}x+\alpha_0) . \end{aligned} \tag{A8}$$

[1] J. von Neumann, *Die Mathematische Grundlagen der Quantenmechanik* (Springer-Verlag, Berlin, 1932).
[2] *Quantum Theory and Measurement,* edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983); see also L. E. Ballantine, Am. J. Phys. **55**, 785 (1987), in particular Sec. D.
[3] M. Cini, Nuovo Cimento B **73**, 27 (1983).
[4] S. Machida and M. Namiki, Prog. Theor. Phys. **63**, 1457 (1980); **63**, 1833 (1980).
[5] H. Nakazato and S. Pascazio, Phys. Lett. A **156**, 386

(1991).
[6] K. Hepp, Helv. Phys. Acta **45**, 237 (1972).
[7] S. Machida and M. Namiki, in *Proceedings of the First International Symposium on the Foundations of Quantum Mechanics,* edited by S. Kamefuchi *et al.* (Physical Society of Japan, Tokyo, 1986), p. 136; M. Namiki, Found. Phys. **18**, 29 (1988); M. Namiki and S. Pascazio, Found. Phys. Lett. **4**, 203 (1991).
[8] S. Kudaka, S. Matsumoto, and K. Kakazu, Prog. Theor. Phys. **82**, 665 (1989).
[9] An analogous point was made by Cini in Ref. [3], p. 32.
[10] M. Namiki and S. Pascazio, Phys. Lett. A **147**, 430 (1990).
[11] N. N. Bogoliubov and D. V. Shirkov, *Quantum Fields* (Benjamin Cummings, Massachussets, 1983), p. 331.
[12] Incidentally, notice that $\langle n_{\rm av}(t)\rangle \xrightarrow[f\to 0]{} (N/2)(1+\cos 2\alpha_0 - 2f\cos 2\alpha_0)$, which is the result obtained perturbatively in Ref. [5].
[13] M. Namiki and S. Pascazio, Phys. Rev. A **44**, 39 (1991).
[14] W. H. Zurek, Phys. Rev. D **26**, 1862 (1982).