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Decoherence, fluctuations and Wigner function in neutron optics

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Abstract
We analyse the coherence properties of neutron wavepackets, after they have interacted with a phase shifter undergoing different kinds of statistical fluctuation. We give a quantitative (and operational) definition of decoherence and compare it to the standard deviation of the distribution of the phase shifts. We find that in some cases the neutron ensemble is more coherent, even though it has interacted with a wider (i.e. more disordered) distribution of shifts. This feature is independent of the particular definition of decoherence: this is shown by proposing and discussing an alternative definition, based on the Wigner function, that displays a similar behaviour. We briefly discuss the notion of entropy of the shifts and find that, in general, it does not correspond to that of decoherence of the neutron.

Keywords: Decoherence, irreversibility, fluctuations, Wigner function

1. Introduction

Decoherence is an interesting phenomenon, related to the long-standing issue of irreversibility. Nowadays, it discloses challenging perspectives in the light of new technologies and related physical applications. There is a widespread consensus [1–3] about the meaning of decoherence, viewed as the loss of quantum mechanical coherence of a physical system in interaction with other systems ('environment'). However, a *quantitative* definition of decoherence is subtle and involves conceptual pitfalls [4]. In addition, it always depends on the experimental configuration. An interesting quantity in this context is the square of the density matrix [5]. Apart from lacking idempotency for mixed states, this quantity enjoys other interesting features [6], but also yields results which are at variance with naive expectations based on the entropy [4].

In this paper we will consider two different definitions of decoherence: the first is operational and stems from an analysis of the visibility in quantum (as well as classical, as we will see) interference experiments. We stress that these experiments are routinely performed in neutron optics [7–9]. The second definition is based on the idempotency defect of the density matrix and is, in this sense, less operational.

In both cases, decoherence displays an 'anomalous' behaviour, both as a function of the features of the fluctuations and the incoming state. Some concrete examples will be considered and discussed. Our analysis will focus on neutron optics and hinge on an approach based on the analysis of statistical fluctuations [10, 11]. However, since our results are just a consequence of the wave nature of neutrons and their coherence properties, we expect that the same general conclusions be valid for other quantum (and classical) waves.

2. Fluctuations in neutron optics

Let us start our analysis by considering a neutron beam that crosses a Mach–Zehnder interferometer (MZI), as schematically shown in figure 1. A phase shifter Δ is placed in the lower arm of the interferometer and $|\psi_{\rm in}\rangle$ is the initial wavepacket.

We neglect wavepacket dispersion effects, so that the outgoing states in the ordinary and extraordinary channels read

$$\begin{aligned} |\psi_O\rangle &= \tfrac{1}{2}[1 + e^{\frac{i}{\hbar}\hat{p}\Delta}]|\psi_{\rm in}\rangle, \\ |\psi_E\rangle &= \tfrac{1}{2}[1 - e^{\frac{i}{\hbar}\hat{p}\Delta}]|\psi_{\rm in}\rangle, \end{aligned} \tag{1}$$

respectively. We focus on the ordinary channel, the analysis for the extraordinary one being identical. Define the operator

$$\hat{O}(\Delta) = \tfrac{1}{2}[1 + e^{\frac{i}{\hbar}\hat{p}\Delta}], \tag{2}$$

that accounts for the state evolution in the ordinary channel,

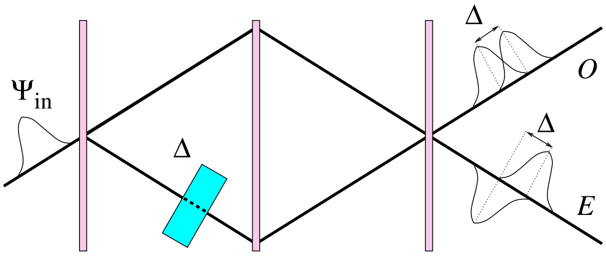

Figure 1. Scheme of an MZI.
(This figure is in colour only in the electronic version)

and consider the output density matrix

$$\rho_O \equiv |\psi_O\rangle\langle\psi_O| = \hat{O}(\Delta)|\psi_{\rm in}\rangle\langle\psi_{\rm in}|\hat{O}(\Delta)^\dagger = \hat{O}(\Delta)\rho_{\rm in}\hat{O}(\Delta)^\dagger, \tag{3}$$

where $\rho_{\rm in}$ is the density matrix of the incoming state. The trace of ρ_O yields the relative frequency of neutrons in the ordinary channel.

Suppose now that the phase shift Δ fluctuates according to a probability law $w(\Delta-\Delta_0)$, Δ_0 being the average phase (operationally defined as the phase that is measured—or inferred [12]—in an interferometric experiment). Therefore one has

$$\int {\rm d}\Delta\, w(\Delta) = 1, \qquad \int {\rm d}\Delta\, w(\Delta)\Delta = 0. \tag{4}$$

The trace of the average density matrix is

$$\mathrm{Tr}\,\overline{\rho_O} = \mathrm{Tr}\int {\rm d}\Delta\, w(\Delta-\Delta_0)\hat{O}(\Delta)\rho_{\rm in}\hat{O}(\Delta)^\dagger = \mathrm{Tr}(\rho_{\rm in}\overline{\hat{O}(\Delta)^\dagger\hat{O}(\Delta)}), \tag{5}$$

where the bar denotes the average over the distribution $w(\Delta-\Delta_0)$. One obtains, after some algebra,

$$\overline{\hat{O}(\Delta)^\dagger\hat{O}(\Delta)} = \frac{1}{2}\left(1+\overline{\cos\frac{\hat{p}\Delta}{\hbar}}\right). \tag{6}$$

Consider now the Fourier transform of the probability density of the fluctuations

$$\Omega(p) \equiv \int {\rm d}\Delta\, w(\Delta){\rm e}^{\frac{\rm i}{\hbar}p\Delta} = \int {\rm d}\Delta\, w(\Delta)\cos\frac{p\Delta}{\hbar} + {\rm i}\int {\rm d}\Delta\, w(\Delta)\sin\frac{p\Delta}{\hbar} = C(p)+{\rm i}S(p), \tag{7}$$

where C and S are respectively the real and the imaginary part of Ω

$$C(p) = \mathrm{Re}\,\Omega(p), \tag{8}$$

$$S(p) = \mathrm{Im}\,\Omega(p). \tag{9}$$

In equation (6) we can write

$$\overline{\cos\frac{\hat{p}\Delta}{\hbar}} = \int {\rm d}\Delta\, w(\Delta-\Delta_0)\cos\frac{\hat{p}\Delta}{\hbar} = \cos\frac{\hat{p}\Delta_0}{\hbar}C(\hat{p}) - \sin\frac{\hat{p}\Delta_0}{\hbar}S(\hat{p}). \tag{10}$$

In this paper, for simplicity, we will always consider symmetric distribution functions, that is $w(\Delta)=w(-\Delta)$. Therefore

$$S(\hat{p}) = 0, \qquad C(\hat{p}) = \Omega(\hat{p}) \tag{11}$$

and (6) becomes

$$\overline{\hat{O}(\Delta)^\dagger\hat{O}(\Delta)} = \frac{1}{2}\left[1+\Omega(\hat{p})\cos\frac{\hat{p}\Delta_0}{\hbar}\right]. \tag{12}$$

We notice, incidentally, that the same results are obtained with a different set-up [8]: consider a polarized neutron that interacts with a magnetic field perpendicular to its spin. Due to the longitudinal Stern–Gerlach effect [13], its wavepacket is split into two components that travel with different speeds and are therefore separated in space. After a projection onto the initial spin state, the resulting final state is slightly different from that considered in the preceding equations: we need to replace $|\psi_O\rangle$ (and analogously $|\psi_E\rangle$) in (1) with

$$|\psi_O\rangle \longrightarrow |\psi'_O\rangle = \hat{O}'(\Delta)|\psi_{\rm in}\rangle, \tag{13}$$

where

$$\hat{O}'(\Delta) = \tfrac{1}{2}[{\rm e}^{-\frac{\rm i}{2\hbar}\hat{p}\Delta} + {\rm e}^{\frac{\rm i}{2\hbar}\hat{p}\Delta}], \tag{14}$$

Δ being in this case the spatial separation between the two wavepackets corresponding to the two spin components. By averaging over Δ it is easy to show that one obtains again (12).

By plugging the average operator (12) into (5) one finally gets

$$\mathrm{Tr}\;\overline{\rho_O} = \frac{1}{2}\left[1+\left\langle\Omega(\hat{p})\cos\frac{\hat{p}\Delta_0}{\hbar}\right\rangle\right], \tag{15}$$

where $\langle\cdots\rangle = \mathrm{Tr}[\rho_{\rm in}\cdots]$ denotes the expectation value over the initial state $\rho_{\rm in}$. On the other hand, the momentum distribution is easily shown to be

$$P_O(p) = \langle p|\overline{\rho_O}|p\rangle = \mathrm{Tr}(|p\rangle\langle p|\overline{\rho_O}) = \frac{1}{2}P_{\rm in}(p)\left[1+\Omega(p)\cos\frac{p\Delta_0}{\hbar}\right], \tag{16}$$

where

$$P_{\rm in}(p) = \langle p|\rho_{\rm in}|p\rangle. \tag{17}$$

We now introduce the *visibility* of the interference pattern (in the ordinary channel)

$$\mathcal{V}(p) \equiv \frac{P_O(p)_{\rm MAX} - P_O(p)_{\rm min}}{P_O(p)_{\rm MAX} + P_O(p)_{\rm min}} = |\Omega(p)|, \tag{18}$$

where $P_O(p)_{\rm MAX}$ ($P_O(p)_{\rm min}$) is the maximum (minimum) value assumed by $P_O(p)$ when Δ_0 varies. By the very definition (7), one can verify that $0 \leqslant \mathcal{V}(p) \leqslant 1$. Notice that, according to this definition, the visibility is a function of momentum p and yields a measure of the fringe visibility of a postselected beam of momentum p as a function of the phase shift Δ_0 [8, 15]. Equivalently, it is a measure of the 'local' spectral visibility, under the assumption of a slowly varying wave envelope, and so it corresponds to (the absolute value of) the amplitude of the cosine function in (16). By using (7) and (18), one infers that the visibility is the modulus of the Fourier transform of the distribution of the shifts Δ and is therefore a quantity that is closely related to the physical features of the phase shifter. In this way we can easily relate the visibility of the interference pattern (and, as we will see below, the decoherence) to the 'environmental' fluctuations. Note that a completely equivalent definition of the spectral visibility (18), which is nevertheless more symmetric and also makes use of the extraordinary channel, reads

$$\mathcal{V}(p) = \max_{\Delta_0} \frac{|P_O(p) - P_E(p)|}{P_O(p) + P_E(p)} = \max_{\Delta_0} \left| \Omega(p) \cos \frac{p\Delta_0}{\hbar} \right| = |\Omega(p)|, \qquad (19)$$

where the momentum distribution of the extraordinary channel is given by

$$P_E(p) = \frac{1}{2} P_{\text{in}}(p) \left[1 - \Omega(p) \cos \frac{p\Delta_0}{\hbar} \right], \qquad (20)$$

whence $P_O(p) + P_E(p) = P_{\text{in}}(p)$. The spectral visibility in the form (19) leads to a straightforward generalization which is at the basis of an operational definition of decoherence.

3. An operational definition of decoherence

Let us endeavour to give a quantitative definition of decoherence based on the definition of visibility given in the previous section. We start from the relative frequency of particles detected in the ordinary and extraordinary channels

$$\mathcal{N}_O(\Delta_0) = \text{Tr}\,\overline{\rho_O} = \frac{1}{2}\left[1 + \left\langle \Omega(\hat{p}) \cos \frac{\hat{p}\Delta_0}{\hbar} \right\rangle \right],$$
$$\mathcal{N}_E(\Delta_0) = \text{Tr}\,\overline{\rho_E} = \frac{1}{2}\left[1 - \left\langle \Omega(\hat{p}) \cos \frac{\hat{p}\Delta_0}{\hbar} \right\rangle \right]. \qquad (21)$$

Their difference is

$$\mathcal{N}_O(\Delta_0) - \mathcal{N}_E(\Delta_0) = \left\langle \Omega(\hat{p}) \cos \frac{\hat{p}\Delta_0}{\hbar} \right\rangle \qquad (22)$$

and one can define a *generalized visibility*

$$\mathcal{V} = \max_{\Delta_0} |\mathcal{N}_O(\Delta_0) - \mathcal{N}_E(\Delta_0)| = \max_{\Delta_0} \left| \left\langle \Omega(\hat{p}) \cos \frac{\hat{p}\Delta_0}{\hbar} \right\rangle \right| = \max_{\Delta_0} \left| \int \mathrm{d}p\, P_{\text{in}}(p) \Omega(p) \cos \frac{p\Delta_0}{\hbar} \right|. \qquad (23)$$

It is apparent that (23) is the straightforward generalization of the spectral visibility (19), because obviously $\mathcal{N}_O + \mathcal{N}_E = 1$. It represents a *global* feature of the outgoing state, in contrast with the local character of (19). Notice, however, that when $P_{\text{in}}(p') = \delta(p' - p)$ (incoming monochromatic beam of momentum p), the generalized visibility (23) reduces to the standard 'local' visibility (18)

$$\mathcal{V} = \max_{\Delta_0} \left| \int \mathrm{d}p'\, \delta(p' - p) \Omega(p') \cos \frac{p'\Delta_0}{\hbar} \right| = \max_{\Delta_0} \left| \Omega(p) \cos \frac{p\Delta_0}{\hbar} \right| = \mathcal{V}(p). \qquad (24)$$

This is a consistency check, because a spectral postselection is equivalent to injecting an incoming monochromatic beam. It is worth noting that the above-mentioned 'monochromatic beam' of momentum p, $P_{\text{in}}(p') = \delta(p' - p)$, is not a plane wave: for example, it can be obtained from the $\delta \to 0$ limit of the *bona fide* density matrix

$$\rho_\delta(p', p'') = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-\frac{(p' - p)^2 + (p'' - p)^2}{4\delta^2} \right]. \qquad (25)$$

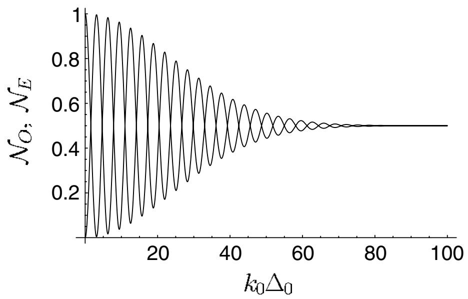

Figure 2. Relative frequencies of neutrons detected in the ordinary $\mathcal{N}_O$ and extraordinary $\mathcal{N}_E$ channel versus $k_0\Delta_0$ ($k_0 = p_0/\hbar$), for an incoming Gaussian wavepacket (29) with $k_0\delta = 12$ and a fluctuation-free phase shifter. The two intensities differ in phase by π and their sum is unity. The generalized visibility (23) is unity.

In general one gets

$$\mathcal{V} \leqslant \max_{\Delta_0} \int \mathrm{d}p\, P_{\text{in}}(p) |\Omega(p)| \left| \cos \frac{p\Delta_0}{\hbar} \right| = \int \mathrm{d}p\, P_{\text{in}}(p) \mathcal{V}(p) = \langle \mathcal{V}(\hat{p}) \rangle. \qquad (26)$$

The generalized visibility yields the maximum 'distance' between the intensities $\mathcal{N}_O$ and $\mathcal{N}_E$ and is bounded by the 'local' visibility averaged over the momentum distribution of the incoming state.

Notice that $P_{\text{in}}(p)$ is a non-negative quantity, while $\Omega(p)$, being a Fourier transform, is not. For this reason, in general, $\max_{\Delta_0}$ does not simply enter the integral in (23), so that (26) is a strict inequality. However, in the particular case $\Omega(p) \geqslant 0$, $\mathcal{V}$ saturates its upper bound, the equality sign holds in (26) and equation (23) simplifies into

$$\mathcal{V} = \int \mathrm{d}p\, P_{\text{in}}(p) \Omega(p) \qquad (\Omega(p) \geqslant 0). \qquad (27)$$

As is probably to be expected, the most interesting cases are those situations in which $\Omega(p)$ is *not* always positive, giving rise to 'anomalous' situations.

In order to understand the physical meaning of the generalized visibility, it is useful to look at the example of a fluctuation-free phase shifter, $w(\Delta) = \delta(\Delta)$, for which (7) yields $\Omega(p) = 1$, so that the generalized visibility (23) becomes

$$\mathcal{V} = \max_{\Delta_0} \left| \int \mathrm{d}p\, P_{\text{in}}(p) \cos \frac{p\Delta_0}{\hbar} \right| = \int \mathrm{d}p\, P_{\text{in}}(p) = 1, \qquad (28)$$

for any incoming distribution P_{in}. This result also follows directly from (27). For instance, for an incoming Gaussian wavepacket

$$P_{\text{in}}(p) = \sqrt{\frac{2\delta^2}{\hbar^2\pi}} \exp\left(-\frac{2\delta^2}{\hbar^2} (p - p_0)^2 \right), \qquad (29)$$

one gets the interference patterns $\mathcal{N}_O$ and $\mathcal{N}_E$ shown in figure 2, where it is apparent that $\mathcal{V} = 1$.

If, on the other hand, the phase shifter fluctuates, the amplitude of the envelope function decreases and $\mathcal{V} < 1$. We therefore give an operational definition of decoherence, by defining a *decoherence parameter*:

$$\varepsilon \equiv 1-\mathcal{V} = 1-\max_{\Delta_0}\left|\left\langle \Omega(\hat{p})\cos\frac{\hat{p}\Delta_0}{\hbar}\right\rangle\right|$$
$$= 1-\max_{\Delta_0}\left|\int \mathrm{d}p\, P_{\mathrm{in}}(p)\Omega(p)\cos\frac{p\Delta_0}{\hbar}\right|. \qquad (30)$$

Notice that, by equation (28), $\varepsilon = 0$ for a fluctuation-free phase shifter (quantum coherence perfectly preserved), while $\varepsilon \to 1$ when the magnitude of the fluctuations increases, $\Omega(p) \to 0$ and the envelope function in figure 2 squeezes away all oscillations, eventually yielding $\mathcal{N}_O(\Delta_0) = \mathcal{N}_E(\Delta_0)$, independently of Δ_0. Observe also that $\mathcal{V}$ and ε are independent of the 'spectral' coherence of the initial state, namely, they do not depend on the off-diagonal terms of the initial density matrix in the momentum basis. It is worth mentioning, in this respect, that this is a feature of our 'interferometric' definition of coherence: interference is essentially a Fourier transform and cannot distinguish a pure state from a mixture with the same momentum distribution [14]. On the other hand, $\mathcal{V}$ and ε strongly depend on the momentum distribution of the initial state (17). In this sense they measure the *loss* of quantum coherence caused by a given physical set-up, independently of the coherence of the incoming state.

It is important to stress that the above definition of decoherence is *operational*. One first measures the relative frequencies of neutrons detected in the ordinary and extraordinary channels as a function of Δ_0, both being measurable quantities. Then one evaluates (23) and computes ε.

4. Some examples

The decoherence parameter (30) depends on the product of the momentum distribution of the incoming beam times the spectrum of the phase-shifter fluctuations, $P_{\mathrm{in}}(p) \times \Omega(p)$. These two ingredients affect ε at the same level. Therefore, their role can be interchanged: by maintaining their product unaltered, there exist 'dual' situations that give exactly the same decoherence parameter with very different kinds of statistical fluctuation and incoming state.

By keeping the above remark in mind, it is interesting to look at some particular cases that can be treated analytically. Let the phases be distributed according to a Gaussian law with standard deviation σ

$$w(\Delta-\Delta_0) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\Delta-\Delta_0)^2}{2\sigma^2}\right), \qquad (31)$$

so that $\Omega(p) = \exp(-p^2\sigma^2/2\hbar^2)$ and the decoherence parameter reads

$$\varepsilon = 1-\max_{\Delta_0}\left|\int \mathrm{d}p\, P_{\mathrm{in}}(p)\exp\left(-\frac{p^2\sigma^2}{2\hbar^2}\right)\cos\left(\frac{p\Delta_0}{\hbar}\right)\right|$$
$$= 1-\int \mathrm{d}p\, P_{\mathrm{in}}(p)\exp\left(-\frac{p^2\sigma^2}{2\hbar^2}\right). \qquad (32)$$

For the Gaussian wavepacket (29) one gets

$$\varepsilon = 1-\sqrt{\frac{\delta^2}{\delta^2+\sigma^2/4}}\exp\left(-\frac{\delta^2}{\delta^2+\sigma^2/4}\frac{\sigma^2k_0^2}{2}\right), \qquad (33)$$

with $k_0 = p_0/\hbar$. This is exact and is shown in figure 3.

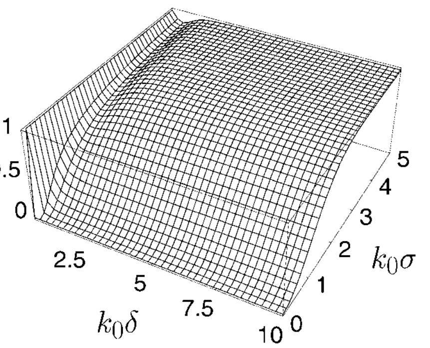

Figure 3. Decoherence parameter ε (33) versus width δ of the Gaussian wavepacket and standard deviation σ of the fluctuating shifts ($k_0 = p_0/\hbar$).

At fixed δ the decoherence parameter (33) increases with σ, although the details of its behaviour are strongly dependent on the spatial width of the packet δ. This behaviour is in agreement with expectation: decoherence ε increases with the magnitude σ of the fluctuations.

For a monochromatic beam ($P_{\mathrm{in}}(p') = \delta(p-p')$)

$$\varepsilon_k = 1-\mathrm{e}^{-\frac{k^2\sigma^2}{2}}, \qquad (34)$$

with $k = p/\hbar$. This is shown in figure 4(a) and can be obtained from (33) in the $\delta \to \infty$ limit. Notice that high momenta are more fragile against fluctuations [11]. Moreover, when the distribution of the shifts is Gaussian, ε_k and equivalently $\mathcal{V}(p)$ are monotonic functions: they both depend 'smoothly' on σ.

Now let the phase shifts be distributed according to the law [4]

$$w(\Delta-\Delta_0) = \frac{1}{\pi}\frac{1}{\sqrt{2\sigma^2-(\Delta-\Delta_0)^2}}, \qquad (35)$$

for $|\Delta-\Delta_0| \leqslant \sqrt{2}\sigma$ and 0 otherwise, with standard deviation $(\int \Delta^2 w(\Delta)\,\mathrm{d}\Delta)^{1/2} = \sigma$. From an experimental perspective this is more convenient and easier to reproduce than the Gaussian distribution (31): indeed, (35) follows from a phase $\Delta(t) = \Delta_0 + \sqrt{2}\sigma\sin t$, where t ('time') is a parameter, uniformly distributed between 0 and 2π, namely $w(\Delta) = \int_0^{2\pi}\mathrm{d}t\,\delta(\Delta-\sqrt{2}\sigma\sin t)/2\pi$. (One can require $\sqrt{2}\sigma \leqslant \Delta_0$, in order that $\Delta(t)$ be positive—and the term $\sqrt{2}\sigma\sin t$ be regarded as a 'small' fluctuation around the average value. However, strictly speaking, this is not necessary from a mathematical point of view.) From (35) and (7) one gets

$$\Omega(p) = \int_{-\sqrt{2}\sigma}^{\sqrt{2}\sigma}\frac{\mathrm{d}\Delta}{\pi}\frac{\mathrm{e}^{\mathrm{i}\frac{p\Delta}{\hbar}}}{\sqrt{2\sigma^2-\Delta^2}}$$
$$= \int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}t}{\pi}\exp\left(\mathrm{i}\frac{\sqrt{2}p\sigma}{\hbar}\sin t\right) = J_0\left(\frac{\sqrt{2}p\sigma}{\hbar}\right), \qquad (36)$$

where J_0 is the Bessel function of order zero. The decoherence parameter (30) reads

$$\varepsilon = 1-\max_{\Delta_0}\left|\int \mathrm{d}p\, P_{\mathrm{in}}(p)J_0\left(\frac{\sqrt{2}p\sigma}{\hbar}\right)\cos\left(\frac{p\Delta_0}{\hbar}\right)\right| \qquad (37)$$

and for a monochromatic beam one obtains ($k = p/\hbar$)

$$\varepsilon_k = 1-\max_{\Delta_0}\left|J_0\left(\frac{\sqrt{2}p\sigma}{\hbar}\right)\cos\left(\frac{p\Delta_0}{\hbar}\right)\right| = 1-|J_0(\sqrt{2}k\sigma)|. \qquad (38)$$

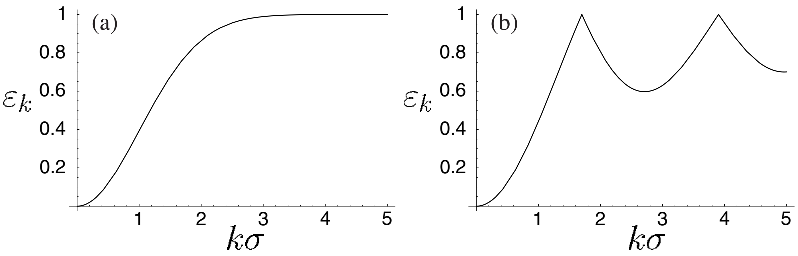

Figure 4. (a) Decoherence parameter ε_k (34) versus $k\sigma$, for a monochromatic beam interacting with a shifter fluctuating according to (31); (b) decoherence parameter ε_k (38) versus $k\sigma$, for a monochromatic beam interacting with a shifter fluctuating according to (35).

This function is shown in figure 4 (b): observe that decoherence is *not* a monotonic function of the noise σ in (35).

A comparison between figures 4(a) and (b) is interesting. In both cases one observes fragility at high momenta $p = \hbar k$. However, the behaviour of decoherence in figure 4(b) is somewhat anomalous and against naive expectation. For a given k, there are situations where the decoherence ε *decreases by increasing* the strength of the fluctuations σ. Note also that we are considering incoming monochromatic beams, whence, according to (24) and (30), $\epsilon_k = 1-\mathcal{V}(\hbar k)$ and the decoherence parameter is strictly related to the standard visibility of the interference pattern. Therefore, in the anomalous regions, one observes an increase in visibility by increasing the fluctuations of the phase shifter, a phenomenon somewhat similar to stochastic resonance [16]. This is true not only for monochromatic beams, but also for narrow distributions (packets) in momentum space.

These anomalous results are not entirely surprising, if one compares them to other known results in classical optics. We will therefore recall in the next section some notions related to the visibility of a classical interference experiment: the visibility can be expressed as the Fourier transform of the spectral distribution of a quasi-monochromatic light source and it displays some 'anomalies' even in cases that are different from our 'Gaussian' example (29).

5. A classical analogy

The phenomena analysed in the previous sections have an interesting classical counterpart that is worth looking at in some detail. In this section we will examine the behaviour of the visibility in a two-beam interference experiment, in relation to the spectral density distribution of the source. We follow Born and Wolf [17]. Suppose we have two beams whose optical difference is $\Delta\mathcal{S}$ and whose wavenumber is $k = 2\pi/\lambda$. Their phase difference reads

$$\delta(k, \Delta\mathcal{S}) = k\Delta\mathcal{S}, \tag{39}$$

and, assuming that they have the same intensity $\mathrm{i}(k)\,\mathrm{d}k$ in the range $[k, k + \mathrm{d}k]$, the intensity at the screen due to the elementary wavenumber range $\mathrm{d}k$ reads

$$\mathrm{i}(k, \Delta\mathcal{S})\,\mathrm{d}k = 2\mathrm{i}(k)[1 + \cos(k\Delta\mathcal{S})]\,\mathrm{d}k. \tag{40}$$

Observe that the different spectral components add incoherently, so

$$I(\Delta\mathcal{S}) = 2\int \mathrm{d}k\,\mathrm{i}(k)[1 + \cos(k\Delta\mathcal{S})] \tag{41}$$

is the intensity at the screen as a function of $\Delta\mathcal{S}$, due to both interfering beams. The quantity k is to be compared to the phase Δ in section 2.

In some cases one deals with light sources that emit with characteristic spectral lines. If we consider only one of these spectral lines, $\mathrm{i}(k)$ is different from zero only in a very small range of k about some mean value k_0. Putting

$$j(k) = 2\mathrm{i}(k_0 + k), \tag{42}$$

the intensity at the screen (41) becomes

$$\begin{aligned} I(\Delta\mathcal{S}) &= \int \mathrm{d}k\; j(k)\{1 + \cos[(k_0 + k)\Delta\mathcal{S}]\} \\ &= N[1 + C(\Delta\mathcal{S})\cos(k_0\Delta\mathcal{S}) - S(\Delta\mathcal{S})\sin(k_0\Delta\mathcal{S})], \end{aligned} \tag{43}$$

where N is a normalization factor, defined as the sum of both the (equal) intensities of the beams, and C and S are the average values of the spectral distribution $j(k)$ of $\cos(k\Delta\mathcal{S})$ and $\sin(k\Delta\mathcal{S})$ respectively

$$N = \int \mathrm{d}k\; j(k), \tag{44}$$

$$C(\Delta\mathcal{S}) = \frac{1}{N}\int \mathrm{d}k\; j(k)\cos(k\Delta\mathcal{S}), \tag{45}$$

$$S(\Delta\mathcal{S}) = \frac{1}{N}\int \mathrm{d}k\; j(k)\sin(k\Delta\mathcal{S}), \tag{46}$$

i.e. C and S are respectively the real and the imaginary part of the Fourier transform $\Omega(\Delta\mathcal{S})$ of $j(k)/N$

$$C(\Delta\mathcal{S}) = \mathrm{Re}\,\Omega(\Delta\mathcal{S}), \qquad S(\Delta\mathcal{S}) = \mathrm{Im}\,\Omega(\Delta\mathcal{S}), \tag{47}$$

$$\Omega(\Delta\mathcal{S}) = \int \mathrm{d}k\,\frac{j(k)}{N}\mathrm{e}^{\mathrm{i}k\Delta\mathcal{S}}. \tag{48}$$

From (43), the intensity at the screen can be written as

$$I = N[1 + |\Omega(\Delta\mathcal{S})|\cos(k_0\Delta\mathcal{S} + \varphi(\Delta\mathcal{S}))], \tag{49}$$

where $\tan\varphi(\Delta\mathcal{S}) = S(\Delta\mathcal{S})/C(\Delta\mathcal{S})$. A comparison with equation (16) shows that $\Delta\mathcal{S}$ plays the same role of p.

Because $j(k)$ is very peaked about $k = 0$, variations of C and S can be considered negligible compared with $\cos(k_0\Delta\mathcal{S})$ and $\sin(k_0\Delta\mathcal{S})$ in equation (43); analogously for φ in (49). Consequently, under the assumption of slowly varying envelope, one can define a 'local' visibility [17], given by

$$\mathcal{V}(\Delta\mathcal{S}) = \frac{I(\Delta\mathcal{S})_{\mathrm{MAX}} - I(\Delta\mathcal{S})_{\mathrm{min}}}{I(\Delta\mathcal{S})_{\mathrm{MAX}} + I(\Delta\mathcal{S})_{\mathrm{min}}} = |\Omega(\Delta\mathcal{S})|, \tag{50}$$

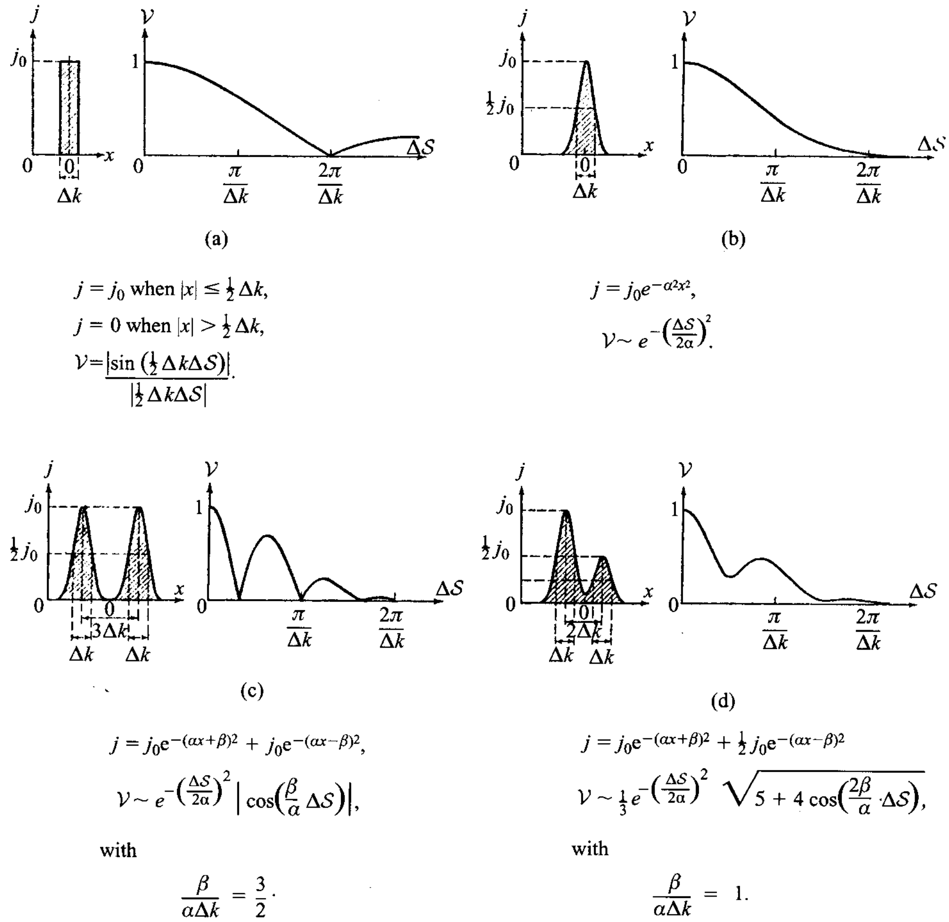

Figure 5. Visibility versus optical path difference $\Delta\mathcal{S}$, for a quasi-monochromatic light source with spectral distribution $j(x)$ (42). Different spectral distribution shapes are considered: squarelike (a), Gaussian (b) and double Gaussian with peaks at the same level (c) and at different levels (d). In (b), (c) and (d) $\Delta k = 2\sqrt{\ln 2}/\alpha$. (Reproduced with permission from [17].)

expressed as a function of the optical path difference $\Delta\mathcal{S}$. The visibility is therefore the amplitude of the cosine function in equation (49). Observe that, whenever $j(k)$ is an even spectral distribution,

$$\mathcal{V}(\Delta\mathcal{S}) = |C(\Delta\mathcal{S})| = |\Omega(\Delta\mathcal{S})| \tag{51}$$

and it is possible to determine (apart from the sign) the Fourier transform of $j(k)/N$ from the visibility.

Equations (18) and (50) are easily compared. The distribution of the phase shifts in the quantum case is replaced by the spectral distribution of the incoherent light source in the classical case. Indeed, $\Omega(\Delta\mathcal{S})$, $C(\Delta\mathcal{S})$ and $S(\Delta\mathcal{S})$ in (47) and (48) correspond to $\Omega(p)$, $C(p)$ and $S(p)$ in (7)–(9), i.e. the Fourier transform of $j(k)/N$ corresponds to that of $w(\Delta)$ (notice that $w(\Delta)$ is normalized to unity).

The visibility curves (51) are shown in figure 5 for different shapes of the spectral distribution $j(k)$. As one can see, they show different behaviour. In figure 5(a) a squarelike spectral distribution gives rise to a visibility function $|\sin y/y|$, in figure 5(b) a Gaussian spectral distribution produces a Gaussian visibility function and in figures 5(c) and (d) two 'double-Gaussian' distributions (with peaks that have or do not have the same level, respectively) yield more complicated visibility functions. Only in case (b), i.e. with a Gaussian spectral distribution, is the visibility a monotonic function of the optical path difference $\Delta\mathcal{S}$. In such a case, the naive expectation is confirmed that, by increasing the optical path difference, the visibility decreases. This is not true in cases (a), (c) and (d), where the visibility is not a decreasing function for every range of $\Delta\mathcal{S}$, and there are regions on the screen where, by increasing the optical path difference, the two-beam interference visibility increases.

Similar results can be obtained if one considers two-beam interference with extended monochromatic light sources. In such a case, the source is treated as a collection of monochromatic pointlike sources that add incoherently and, instead of $j(k)\,\mathrm{d}k$, one deals with $\mathrm{i}(\alpha)\,\mathrm{d}\alpha$, the elementary intensity due to such pointlike sources of angular width $\mathrm{d}\alpha$. As a result, the visibility is related to the normalized Fourier transform of the extended source angular intensity distribution. This problem was studied as early as the end of the 19th century [18] and led Michelson to the construction of his stellar interferometer [19].

6. Wigner function in the ordinary channel

In the previous sections we have proposed a definition of decoherence based on the visibility of the quantum interference pattern. As we have seen, this definition has some unexpected features, at variance with expectation. We also found an analogy in classical optics. However, alternative definitions of decoherence are possible, based on the density matrix and on the Wigner function. Let us therefore briefly recall the definition and some properties of the Wigner function.

The Wigner quasidistribution function [20] can be defined in terms of the density matrix ρ as

$$W(x,k) = \frac{1}{2\pi}\int \mathrm{d}\xi\, \mathrm{e}^{-\mathrm{i}k\xi}\langle x+\xi/2|\rho|x-\xi/2\rangle, \qquad (52)$$

where x and $p = \hbar k$ are the position and momentum of the particle. One easily checks that the Wigner function is normalized to unity and its marginals represent the position and momentum distributions

$$\mathrm{Tr}\,\rho = \int \mathrm{d}x\,\mathrm{d}k\, W(x,k) = 1, \qquad (53)$$

$$P(x) = \langle x|\rho|x\rangle = \int \mathrm{d}k\, W(x,k), \qquad (54)$$

$$P(k) = \langle k|\rho|k\rangle = \int \mathrm{d}x\, W(x,k). \qquad (55)$$

The analyses of the properties of quantum states based on the Wigner function are useful because they enable one to make prompt comparisons with fields like quantum optics [21] and quantum tomography [22].

We focus on one-dimensional systems and assume that the wavefunction is well approximated by a Gaussian

$$\psi(x) = \langle x|\psi\rangle = \frac{1}{(2\pi\delta^2)^{1/4}}\exp\left[-\frac{(x-x_0)^2}{4\delta^2} + \mathrm{i}k_0 x\right], \qquad (56)$$

$$\phi(k) = \langle k|\psi\rangle = \frac{1}{(2\pi\delta_k^2)^{1/4}}\exp\left[-\frac{(k-k_0)^2}{4\delta_k^2} - \mathrm{i}(k-k_0)x_0\right]$$
$$= \left(\frac{2\delta^2}{\pi}\right)^{1/4}\exp\left[-\delta^2(k-k_0)^2 - \mathrm{i}(k-k_0)x_0\right], \qquad (57)$$

where $\psi(x)$ and $\phi(k)$ are the wavefunctions in the position and momentum representation, respectively, δ is the spatial spread of the wavepacket, $\delta_k\delta = 1/2$, x_0 is the initial average position of the particle and $p_0 = \hbar k_0$ its average momentum. The two functions above are both normalized to unity. The Wigner function for the state (56) and (57) is readily calculated

$$W(x,k) = \frac{1}{\pi}\exp\left[-\frac{(x-x_0)^2}{2\delta^2}\right]\exp[-2\delta^2(k-k_0)^2]. \quad (58)$$

Consider now a neutron wavepacket that is split and then recombined in an interferometer, with a phase shifter Δ placed in one of the two routes. The Wigner function in the ordinary channel (transmitted component) is readily computed:

$$W_O(x,k,\Delta) = \frac{1}{4\pi}\exp[-2\delta^2(k-k_0)^2]$$
$$\times\left[\exp\left(-\frac{(x-x_0+\Delta)^2}{2\delta^2}\right) + \exp\left(-\frac{(x-x_0)^2}{2\delta^2}\right)\right.$$
$$\left. + 2\exp\left(-\frac{(x-x_0+\frac{\Delta}{2})^2}{2\delta^2}\right)\cos(k\Delta)\right]. \qquad (59)$$

Notice that, for $\Delta \neq 0$, it is not normalized to unity (some neutrons end up in the extraordinary channel—reflected component) and that for $\Delta = 0$ (no phase shifter) one recovers (58).

7. Alternative definition of decoherence

We look at a particular case and assume that the shifts Δ fluctuate around their average Δ_0 according to the Gaussian law (31). The average Wigner function reads

$$\overline{W}(x,k) = \int \mathrm{d}\Delta\, w(\Delta) W(x,k,\Delta) \qquad (60)$$

and represents a partially mixed state. Essentially, this Wigner function represents the whole ensemble of neutrons in an experimental run. For the double-Gaussian state (59), obtained when a neutron beam crosses an interferometer, the average Wigner function in the ordinary channel reads

$$\overline{W}_O(x,k) = \frac{\exp[-2\delta^2(k-k_0)^2]}{4\pi}$$
$$\times\left\{\exp\left[-\frac{x^2}{2\delta^2}\right] + \sqrt{\frac{\delta^2}{\delta^2+\sigma^2}}\exp\left[-\frac{(x+\Delta_0)^2}{2(\delta^2+\sigma^2)}\right]\right.$$
$$+ 2\sqrt{\frac{\delta^2}{\delta^2+\frac{\sigma^2}{4}}}\exp\left[-\frac{(x+\frac{\Delta_0}{2})^2 + k^2\delta^2\sigma^2}{2(\delta^2+\frac{\sigma^2}{4})}\right]$$
$$\left.\times\cos\left(k\frac{2\delta^2\Delta_0 - x\sigma^2}{2(\delta^2+\frac{\sigma^2}{4})}\right)\right\}, \qquad (61)$$

where we set $x_0 = 0$ for simplicity. Its momentum marginal (55) (momentum distribution function) can be computed analytically and is of interest, because it displays fragility at high momenta [8, 11]:

$$P(k) = \sqrt{\frac{\delta^2}{2\pi}}\exp[-2\delta^2(k-k_0)^2]$$
$$\times\left[1+\exp\left(-\frac{k^2\sigma^2}{2}\right)\cos(k\Delta_0)\right]. \qquad (62)$$

The average Wigner function (61) is shown in figure 6. One clearly observes a strong (exponential) suppression of interference at high values of k. Notice that the oscillating part of the Wigner function depends on x. This is due to the fact that the statistical fluctuations are localized in only one of the two routes of the interferometer.

The loss of quantum coherence is clearly visible in figure 6 as the level of noise σ increases. One can try to corroborate this qualitative conclusion by introducing a quantitative notion of decoherence based on the Wigner function; however, as we shall see in a while, one runs into the same kind of difficulty as encountered in section 4. We first recall that there is an interesting relation between the square of the Wigner function and the square of the density matrix:

$$\int \mathrm{d}x\,\mathrm{d}k\, W(x,k)^2 = \frac{\mathrm{Tr}\,\rho^2}{2\pi}. \qquad (63)$$

It is therefore possible to define an alternative decoherence parameter [4], that takes into account the coherence properties of the neutron ensemble (we drop the average-bar on ρ and W for simplicity)

$$\varepsilon = 1 - \frac{\mathrm{Tr}\,\rho^2}{(\mathrm{Tr}\,\rho)^2} = 1 - \frac{2\pi\int \mathrm{d}x\,\mathrm{d}k\, W(x,k)^2}{\left(\int \mathrm{d}x\,\mathrm{d}k\, W(x,k)\right)^2}. \qquad (64)$$

This quantity measures the degree of 'purity' of a quantum state: it is maximum when the state is maximally mixed

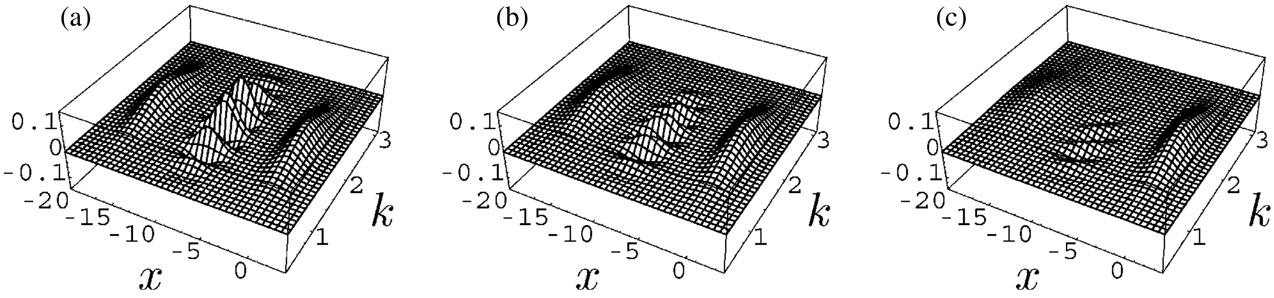

Figure 6. Wigner function in the ordinary channel (61) for different values of the standard deviation σ in (31). From left to right, $\sigma = 0, 0.9$ and 1.8 Å. We set $x_0 = 0$, $k_0 = 1.7$ Å^{-1}, $\delta = 1.1$ Å and $\Delta_0 = 16.1$ Å. Position x and momentum k are measured in Å and Å^{-1}, respectively. Notice the strong suppression of interference at large values of momentum, both in (b) and (c). The interference term in (61) (oscillating part of the Wigner function) depends on x.

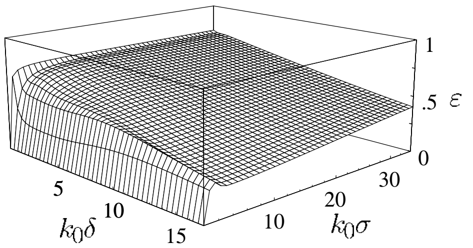

Figure 7. Decoherence parameter versus coherence length of the wavepacket δ and standard deviation of the fluctuation σ for a double-Gaussian wavepacket (59) in the ordinary channel of an MZI. We set $k_0\Delta_0 = 27.4$. The decoherence parameter is not a monotonic function of σ for every value of δ. Notice that ε never reaches unity ($\varepsilon \leqslant 3/4$): this is due to the fact that only one Gaussian (in one route of the interferometer) undergoes statistical fluctuations (see equation (61) and figure 6).

($\mathrm{Tr}\,\rho^2 < \mathrm{Tr}\,\rho$) and vanishes when the state is pure ($\mathrm{Tr}\,\rho^2 = \mathrm{Tr}\,\rho$): in the former case the fluctuations of Δ are large and the quantum mechanical coherence is completely lost, while in the latter case Δ does not fluctuate and the quantum mechanical coherence is perfectly preserved. The parameter (64) was introduced within the framework of the so-called 'many Hilbert space' theory of quantum measurements [2] and yields a quantitative estimate of decoherence. The related quantity $\mathrm{Tr}\,\rho - \mathrm{Tr}\,\rho^2$ was first considered by Watanabe [5] in 1939(!). A quantity related to ε was also introduced in order to get a quantitative estimate of information for a quantum system [23].

It is also worth noticing that the notion of decoherence just introduced is based on the square of the density matrix (or Wigner function) and therefore is not accessible to a direct measurement procedure. In this sense, it is less 'operational' than that discussed in section 3.

The decoherence parameter (64) is shown in figure 7 as a function of the coherence length of the wavepacket δ in (56)–(59) and the standard deviation of the fluctuations σ. It is *not* a monotonic function of σ for all values of δ. Once again, as in section 4, there are situations in which a larger noise yields a more coherent wavepacket (according to a given definition). The behaviour of ε has a nontrivial dependence both on the fluctuations (σ) and on the wavepacket properties (k_0 and δ).

8. Entropy

The conclusions of the previous sections can be corroborated and put on a somewhat sounder basis by computing the entropy of the distribution of the shifts according to the formula

$$S = -\int \mathrm{d}\Delta\, w(\Delta) \log w(\Delta). \tag{65}$$

This quantity yields an estimate of the collective 'degree of disorder' of the distribution of the shifts $w(\Delta)$. One can draw general conclusions about the behaviour of S as a function of a parameter σ characterizing the width of the distribution. Indeed, let $w(\Delta; \sigma)$ be the symmetric distribution with the properties (4), σ being its standard deviation. By assuming that the distribution function w depends only on the single-dimensional parameter σ, then it must scale according to

$$w(\Delta; \sigma) = \frac{1}{\sigma} w\left(\frac{\Delta}{\sigma}; 1\right). \tag{66}$$

Therefore

$$\begin{aligned} S(\sigma) &= -\int \mathrm{d}\Delta\, w(\Delta;\sigma) \log w(\Delta;\sigma) \\ &= -\int \frac{\mathrm{d}\Delta}{\sigma} w\left(\frac{\Delta}{\sigma}; 1\right) \log\left[\frac{1}{\sigma} w\left(\frac{\Delta}{\sigma}; 1\right)\right] \\ &= -\int \mathrm{d}\Delta'\, w(\Delta'; 1) \log w(\Delta'; 1) + \int \mathrm{d}\Delta'\, w(\Delta'; 1) \log\sigma \\ &= S(1) + \log\sigma, \end{aligned} \tag{67}$$

where $S(1)$ is independent of σ and depends only on the form of the distribution function. $S(\sigma)$ is clearly an increasing function of σ.

For example, the Gaussian distribution (31) yields [24]

$$S(\sigma) = \log\sigma + \tfrac{1}{2}\log(2\pi e), \tag{68}$$

while the 'sine' distribution (35) yields

$$S(\sigma) = \log\sigma - \tfrac{1}{2}\log 2. \tag{69}$$

Therefore, the behaviour of the decoherence parameter ε as a function of the entropy S of the shifts is qualitatively equivalent to its behaviour as a function of the standard deviation σ. Indeed, figures 3, 4 and 7 would differ only for a logarithmic scale on the abscissae. As we have seen in this paper, in general, the two quantities S and ε do not necessarily agree:

in other words, the loss of quantum mechanical coherence is not necessarily larger when the neutron beam interacts with fluctuating shifts of larger entropy. This is at variance with expectation and is the main result of this paper.

9. Conclusions

We have introduced and discussed some interference experiments that display some 'anomalies' both in the classical and in the quantum domains. A neutron beam partially loses its quantum coherence as a consequence of the fluctuations of the phase shifts Δ. One should emphasize that we have considered the case of 'slow' fluctuations, in the sense that each neutron crosses a phase shifter of length L, but the length of the shifter varies for different neutrons in the beam (different 'events'). We have supposed that every neutron undergoes a shift Δ that is statistically distributed according to a distribution law $w(\Delta)$.

We focused our attention on two alternative decoherence parameters. The first is defined in terms of a generalized visibility of the interference pattern in a double-slit experiment (MZI) and is more operational. The second hinges upon less operational concepts, such as the square of the density matrix (or Wigner function).

All our results corroborate the ideas expressed elsewhere [4] and make it apparent that the concept of loss of quantum mechanical coherence deserves clarification and additional investigation. It would also be interesting to discuss analogies and differences with conceptual experiments in which decoherence is complemented by Welcher–Weg information [25].

Acknowledgment

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