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Emergence of a Wiener process as a result of the quantum mechanical interaction with a macroscopic medium

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Abstract

We analyze a modified version of the Coleman–Hepp model, which is able to take into account energy-exchange processes between the incoming particle and the linear array made up of N spin-$\frac{1}{2}$ systems. We bring to light the presence of a Wiener dissipative process in the weak-coupling, macroscopic ($N \to \infty$) limit. In such a limit and a restricted portion of the total Hilbert space, the particle undergoes a sort of Brownian motion, while the free Hamiltonian of the spin array serves as a Wiener process. No partial trace is computed over the states of the spin system (which plays the role of "reservoir"). The mechanism of appearance of the stochastic process is discussed and contrasted to other noteworthy examples in the literature. The links with van Hove's "$\lambda^2 T$" limits are emphasized.

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1. Introduction

The derivation of a dissipative dynamics in quantum mechanics and quantum field theory is a long-standing problem. A very important contribution to this issue was given by Van Hove [1], who clarified the main features of a quantum dissipative dynamics and was able to derive a master equation from the Schrödinger equation, in an appropriate limit (his famous "$\lambda^2 T$" limit), via the so-called "diagonal singularity". It is important to stress that Van Hove's ansatz replaced Pauli's random-phase assumption [2].

Dissipation in quantum mechanics can emerge as a result of the interaction between a particle and a macroscopic "environment". Many other important contributions

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have been given to this issue, in the attempt to set up a general physical framework. However, unlike in Van Hove's case, a dissipative dynamics is derived under some assumptions for the energy spectrum of the environment system, by computing the partial trace over the states of the latter. This is true for a whole class of interesting models, in which the environment is schematized with a collection of harmonic oscillators [3,4], and one derives dissipative equations for the "object" particle. It is worth observing that the dissipation constant in the Langevin-type equation is derived from the underlying dynamics under some (reasonable) assumptions on the spectrum of the Hamiltonian of the environment system. Of course, the assumptions on the spectrum are integrating part of the specifications of the model chosen. Yet, it would be somewhat preferable and more consistent to start from a specific microscopic model, solve the dynamics and find an appropriate limit in which the model realizes the desired continuous spectrum. Notice also that a "quantum Langevin equation" has some well-known peculiar properties, such as a non-markoffian random force [5] and colored noise [6,7].

In this paper we shall analyze the so-called "AgBr" model [8,9], which has played an important role in the quantum measurement problem. This model is relatively simple, yet extremely interesting from the physical point of view. We shall base our discussion on a modified version [10] of the above model, that is able to take into account energy-exchange processes. The modified AgBr model provides an interesting nontrivial example of realization of Van Hove's diagonal singularity and displays the occurrence of an exponential regime at all times in the weak-coupling, macroscopic limit [11]. In this way, a door is open to investigate the occurrence of a dissipative dynamics and its link with a quantum measurement process. We shall solve the equations of motion, take a weak-coupling, macroscopic limit, and obtain a Wiener process in a restricted portion of the Hilbert space of the total system. This technique is to be contrasted to the computation of a partial trace over the states of the macroscopic system, and will represent the main difference between the present work and other ones, based on partial tracing.

This work has interesting spinoffs for the quantum measurement problem [12]. It is commonly believed that a quantum measurement occurs via a dephasing (decoherence) process [13–16]. Since "decoherence" is nothing but the disappearance of the off-diagonal elements of the density matrix of the quantum system, and since a system described by a diagonal density matrix exhibits a purely stochastic behavior [17], one is led to expect a connection between dissipation, irreversibility and a quantum measurement process [11,18,19]. In this paper, we shall concentrate on a particular aspect of the above-mentioned issues. Our main purpose is to derive a stochastic process from an underlying Hamiltonian dynamics and to clarify in which sense it is possible to identify a Wiener process for a dynamical variable of the microscopic system under investigation.

The plan of the paper is as follows. We review the main properties of the modified AgBr Hamiltonian in Section 2, focusing on those characteristics that hint at the presence of a stochastic process. In Section 3 we obtain the operators in the Heisenberg picture. Finally, we bring to light the stochastic process in Section 4, where we show that there is a Gaussian process of the Wiener type. All these results are exact. Section 5 is devoted to conclusions and comments.

2. Review of the AgBr Hamiltonian

The modified AgBr Hamiltonian [10] describes the interaction between an ultra-relativistic particle Q and a one-dimensional array (D-system), made up of N spin-$\frac{1}{2}$ objects. The array can be viewed as a caricature of a linear "photographic emulsion" of AgBr molecules, if one identifies the *down* (ground) state of the spin with the undivided molecule and the *up* state (whose excitation energy is $\hbar\omega$) with the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential. The total Hamiltonian for the $Q+D$ system reads

$$H = H_0 + H', \qquad H_0 = H_Q + H_D, \tag{2.1}$$

where H_Q and H_D, the free Hamiltonians of the Q particle and H of the "detector" D, respectively, and the interaction Hamiltonian H' are written as

$$H_Q = c\hat{p}, \qquad H_D = \frac{1}{2}\hbar\omega \sum_{n=1}^{N} (1 + \sigma_3^{(n)}),$$
$$H' = \sum_{n=1}^{N} V(\hat{x} - x_n)\left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\hat{x}\right) + \sigma_-^{(n)} \exp\left(+i\frac{\omega}{c}\hat{x}\right)\right]. \tag{2.2}$$

Here $\hat{p}$ is the momentum of the Q particle, $\hat{x}$ its position, V a real potential, $x_n\,(n = 1, \dots, N)$ the positions of the scatterers in the array ($x_n > x_{n-1}$) and $\sigma_{i,\pm}^{(n)}$ the Pauli matrices acting on the nth site. The above Hamiltonian has attracted the attention of several researchers [20] due, in particular, to the presence of the free Hamiltonian of the array H_D, which enables one to distinguish energetically the up and down states and makes the energy-exchange processes between Q and D physically meaningful. The original Hamiltonian [8] is reobtained in the $\omega = 0$ limit.

Let us review the main results obtained from this model [10,11]. The evolution operator in the interaction picture

$$U_I(t,t') = e^{iH_0t/\hbar}e^{-iH(t-t')/\hbar}e^{-iH_0t'/\hbar} = e^{-i\int_{t'}^{t} H_I'(t')dt'/\hbar}, \tag{2.3}$$

where $H_I'(t)$ is the interaction Hamiltonian in the interaction picture, can be computed exactly as

$$U_I(t) \equiv U_I(t,0) = e^{iH_0t/\hbar}e^{-iHt/\hbar}$$
$$= \prod_{n=1}^{N} \exp\left[-\frac{i}{\hbar}\int_0^t dt'\, V(\hat{x} + ct' - x_n)(\sigma_+^{(n)}e^{-i\omega\hat{x}/c} + \text{h.c.})\right], \tag{2.4}$$

and a straightforward calculation yields the S-matrix

$$S^{[N]} = \lim_{\substack{t \to +\infty \\ t' \to -\infty}} U_I(t, t') = \prod_{n=1}^{N} S_{(n)}, \quad S_{(n)} = \exp\left[-i \frac{V_0 \Omega}{\hbar c} \boldsymbol{\sigma}^{(n)} \cdot \boldsymbol{u} \right], \tag{2.5}$$

where $\boldsymbol{u} = [\cos(\omega x/c), \sin(\omega x/c), 0]$ and $V_0 \Omega \equiv \int_{-\infty}^{\infty} V(x)\, dx < \infty$. The above expression enables us to define the "spin-flip" probability, i.e. the probability of dissociating one AgBr molecule:

$$q \equiv \sin^2 \left[\frac{V_0 \Omega}{c\hbar} \right]. \tag{2.6}$$

By defining

$$\alpha_n \equiv \alpha_n(\hat{x}, t) \equiv \int_0^t \frac{dt'}{\hbar} V(\hat{x} + ct' - x_n) \tag{2.7}$$

and

$$\sigma_{\pm}^{(n)}(\hat{x}) \equiv \sigma_{\pm}^{(n)} e^{\mp i \omega \hat{x}/c}, \tag{2.8}$$

which satisfy, together with $\sigma_3^{(n)}$, the SU(2) algebra

$$\begin{aligned} [\sigma_3^{(n)}, \sigma_{\pm}^{(n)}(\hat{x})] &= \pm 2 \delta_{mn} \sigma_{\pm}^{(n)}(\hat{x}), \\ [\sigma_{-}^{(m)}(\hat{x}), \sigma_{+}^{(n)}(\hat{x})] &= -\delta_{mn} \sigma_3^{(n)}, \end{aligned} \tag{2.9}$$

we can return to the Schrödinger picture by inverting Eq. (2.4). We disentangle the exponential in U_I by making use of (2.9) and obtain

$$e^{-iHt/\hbar} = e^{-iH_0 t/\hbar} \prod_{n=1}^{N} \left(e^{-i \tan(\alpha_n) \sigma_{+}^{(n)}(\hat{x})} e^{-\ln \cos(\alpha_n) \sigma_3^{(n)}} e^{-i \tan(\alpha_n) \sigma_{-}^{(n)}(\hat{x})} \right). \tag{2.10}$$

Let us concentrate our attention on the situation in which the Q particle is initially located at $x' < x_1$, where x_1 is the position of the first scatterer in the linear array, and moves toward the array with speed c. The spin system is initially set in the ground state $|0\rangle_N$ of the free Hamiltonian H_D (all spins down). The propagator is defined by

$$G(x, x', t) \equiv {}_N\langle 0, x | e^{-iHt/\hbar} | x', 0 \rangle_N, \tag{2.11}$$

where $|x, 0\rangle_N \equiv |x\rangle \otimes |0\rangle_N$. If we place the spin array at the far right of the origin ($x_1 > 0$) and consider the case in which the potential V has a compact support and the Q particle is initially located at the origin $x' = 0$, we obtain

$$G(x, 0, t) = \delta(x - ct) \prod_{n=1}^{N} \cos \tilde{\alpha}_n(t), \quad \tilde{\alpha}_n(t) \equiv \int_0^t \frac{dt'}{\hbar} V(ct' - x_n). \tag{2.12}$$

Note that, due to the choice of the free Hamiltonian H_Q, the Q wave packet does not disperse and moves with constant speed c. In this paper we shall exclusively consider the weak-coupling, macroscopic limit

$$N \to \infty \quad \text{and} \quad q \simeq \left(\frac{V_0 \Omega}{\hbar c}\right)^2 = O(N^{-1}), \tag{2.13}$$

which is equivalent to the requirement that the total number of spin flips $\bar{n} = qN$ be finite in the macroscopic limit $N \to \infty$. Notice that, if we set

$$x_n = x_1 + (n-1)d, \qquad L = x_N - x_1 = (N-1)d, \tag{2.14}$$

and let $d/L \to 0$ as $N \to \infty$, a summation over n can be replaced by a definite integration according to

$$q \sum_{n=1}^{N} f(x_n) \to \frac{\bar{n}}{L} \int_{x_1}^{x_N} dy\, f(y). \tag{2.15}$$

In this case, by making use of the Fermi–Yang approximation $V(y) = V_0 \Omega \delta(y)$, (2.12) becomes

$$G \propto \exp\left(\sum_{n=1}^{N} \ln\left[\cos \int_0^{ct} dx \frac{V_0 \Omega}{c\hbar} \delta(x - ct) \right] \right)$$

$$\to \exp\left(-\frac{\bar{n}}{2L} [(ct - x_1)\theta(x_N - ct)\theta(ct - x_1) + L\theta(ct - x_N)] \right), \tag{2.16}$$

where the arrow will henceforth denote the weak-coupling, macroscopic limit (2.13), (2.15). The system attains an exponential regime as soon as the interaction starts: Indeed, if $x_1 < ct < x_N$,

$$G \propto \exp\left(-\bar{n} \frac{(ct - x_1)}{2L} \right). \tag{2.17}$$

Notice the absence of the "Gaussian" regime, characterized by a vanishing derivative at $t = 0$ [21,22], and of the power law at long times [23]. This result is valid for the propagator (2.11), which involves position eigenstates of the Q particle. If these are substituted by (normalizable) wave packets, small deviations from the exponential law appear at short times [19], in agreement with general mathematical theorems [21].

The result (2.17) hints at the presence of a dissipative process of some sort, at least in a restricted portion of the Hilbert space of the total ($Q + D$) system. Such a dissipative process was brought to light in Ref. [24], where it was shown that a Wiener process appears in the weak-coupling, macroscopic limit (2.13), (2.15). In the following sections we shall derive all results (including "border effects") in full generality, discuss their meaning and clarify in which sense and under which conditions it is possible to identify the presence of a dissipative process. The important links with Van Hove's $\lambda^2 T$ limit [1] will also be properly emphasized.

3. Heisenberg operators

In order to bring to light the emergence of a dissipative process in the particle-detector interaction, it is essential to study the temporal evolution of all the operators involved in the interaction process. It is therefore convenient to work in the Heisenberg picture. First of all, notice that the total Hamiltonian of the system is constant in time

$$\begin{aligned} H(t) &= c\hat{p}(t) + \frac{\hbar\omega}{2}\sum_n (1 + \sigma_3^{(n)}(t)) \\ &\quad + \sum_n V(\hat{x}(t) - x_n)\,[\sigma_+^{(n)}(t)e^{-i\omega\hat{x}(t)/c} + \sigma_-^{(n)}(t)\,e^{i\omega\hat{x}(t)/c}] \\ &= H(0)\,, \end{aligned} \tag{3.1}$$

where $H(0) \equiv H$ is the total Hamiltonian of the system in the Schrödinger picture. Let us focus our attention on the free Hamiltonian of the Q particle. From (3.1) we get

$$\begin{aligned} c\hat{p}(t) &= c\hat{p}(0) + \frac{\hbar\omega}{2}\sum_n (\sigma_3^{(n)} - \sigma_3^{(n)}(t)) \\ &\quad + \sum_n \{V(\hat{x} - x_n)\,[\sigma_+^{(n)}e^{-i\omega\hat{x}/c} + \text{h.c.}] \\ &\quad - V(\hat{x}(t) - x_n)\,[\sigma_+^{(n)}(t)\,e^{i\omega\hat{x}(t)/c} + \text{h.c.}]\}\,, \end{aligned} \tag{3.2}$$

where $c\hat{p}(0) \equiv c\hat{p}$ and $\sigma_i^{(n)} \equiv \sigma_i^{(n)}(0)$ $(i = 3,\ \pm)$ are operators in the Schrödinger picture. In order to solve Eq. (3.2), we need the explicit forms of the Heisenberg operators $\hat{x}(t)$, $\sigma_3^{(n)}(t)$ and $\sigma_\pm^{(n)}(t)$. To this end, we shall make use of disentanglement formula (2.10).

The calculation of the operator $\hat{x}(t)$ is straightforward and yields

$$\hat{x}(t) = e^{iHt/\hbar}\hat{x}e^{-iHt/\hbar} = \hat{x} + ct\,. \tag{3.3}$$

On the other hand, the evaluation of $\sigma_{i,\pm}^{(n)}(t)$ is more involved; let us first show in full how to calculate the operator $\sigma_+^{(n)}(t)$. With the help of (2.10) this operator can be rewritten as

$$\begin{aligned} \sigma_+^{(n)}(t) &= e^{iHt/\hbar}\sigma_+^{(n)}e^{-iH_0t/\hbar}\prod_m D_m\,, \\ D_m &\equiv e^{-i\tan(\alpha_m)\sigma_+^{(m)}(\hat{x})}e^{-\ln\cos(\alpha_m)\sigma_3^{(m)}}e^{-i\tan(\alpha_m)\sigma_-^{(m)}(\hat{x})}\,. \end{aligned} \tag{3.4}$$

By observing that

$$e^{iHt/\hbar}\sigma_+^{(n)}e^{-iHt/\hbar} = e^{iHt/\hbar}e^{i\omega\hat{x}/c}\sigma_+^{(n)}(\hat{x})e^{-iH_0t/\hbar}$$
$$= e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}\sigma_+^{(n)}(\hat{x})e^{-iH_0t/\hbar} = e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}\sigma_+^{(n)}(\hat{x}), \tag{3.5}$$

Eq. (3.4) becomes

$$e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}[\sigma_+^{(n)}(\hat{x})e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}]\prod_{m\neq n}D_m\,. \tag{3.6}$$

We now evaluate the term in square brackets. By making use of the formulas

$$e^{a\sigma_3^{(n)}}\sigma_+^{(n)}e^{-a\sigma_3^{(n)}} = e^{2a}\sigma_+^{(n)},\quad e^{b\sigma_-^{(n)}}\sigma_+^{(n)}e^{-b\sigma_-^{(n)}} = \sigma_+^{(n)} - b\sigma_3^{(n)} + \frac{b^2}{2!}(-2\sigma_-^{(n)}), \tag{3.7}$$

we obtain

$$\sigma_+^{(n)}(\hat{x})e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}$$
$$= e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{2\ln\cos(\alpha_n)}\sigma_+^{(n)}(\hat{x})\,e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}$$
$$= e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})}e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}}e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}\cos^2\alpha_n$$
$$\times(\sigma_+^{(n)}(\hat{x}) - i\sigma_3^{(n)}\tan\alpha_n - (i\tan\alpha_n)^2\sigma_-(\hat{x}))\,. \tag{3.8}$$

By substituting (3.8) into Eq. (3.6), we get

$$\sigma_+^{(n)}(t) = e^{i\omega(\hat{x}+ct)/c}e^{iHt/\hbar}e^{-iH_0t/\hbar}\prod_m e^{-i\tan(\alpha_m)\sigma_+^{(m)}(\hat{x})}e^{-\ln\cos(\alpha_m)\sigma_3^{(m)}}e^{-i\tan(\alpha_m)\sigma_-^{(m)}(\hat{x})}$$
$$\times\left(\sigma_+^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_-^{(n)}(\hat{x})\sin^2\alpha_n\right)$$
$$= e^{i\omega(\hat{x}+ct)/c}\left(\sigma_+^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_-^{(n)}(\hat{x})\sin^2\alpha_n\right). \tag{3.9}$$

The Hermitian conjugate of (3.9) yields

$$\sigma_-^{(n)}(t) = e^{i\omega(\hat{x}+ct)/c}\left(\sigma_-^{(n)}(\hat{x})\cos^2\alpha_n - \frac{i}{2}\sigma_3^{(n)}\sin 2\alpha_n + \sigma_+^{(n)}(\hat{x})\sin^2\alpha_n\right) \tag{3.10}$$

and by using an analogous procedure, together with the formula

$$e^{b\sigma_+^{(n)}}\sigma_3^{(n)}e^{-b\sigma_+^{(n)}} = (\sigma_3^{(n)} - 2b\sigma_+^{(n)}) \tag{3.11}$$

and its Hermitian conjugate, we get

$$\sigma_3^{(n)}(t) = \sigma_3^{(n)} \cos 2\alpha_n - i(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x})) \sin 2\alpha_n \,. \tag{3.12}$$

The results (3.9), (3.10) and (3.12) and the relation

$$\sigma_\pm^{(n)}(t) e^{\mp i\omega(\hat{x}+ct)/c} = \sigma_\pm^{(n)} e^{\mp i\omega\hat{x}/c} = \sigma_\pm^{(n)}(\hat{x}) \,, \tag{3.13}$$

lead us to the final expression

$$\begin{aligned} c\hat{p}(t) = {} & c\hat{p}(0) + \hbar\omega \sum_n \sigma_3^{(n)} \sin^2 \alpha_n(\hat{x}, t) + i\frac{\hbar\omega}{2} \sum_n [(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x})) \sin 2\alpha_n(\hat{x}, t)] \\ & + \sum_n [V(\hat{x} - x_n) - V(\hat{x} + ct - x_n)] (\sigma_+^{(n)}(\hat{x}) + \sigma_-^{(n)}(\hat{x})) \,. \end{aligned} \tag{3.14}$$

All the above results are *exact*.

4. The stochastic process

Having found the explicit (and exact) expressions for the quantum operators of our system in the Heisenberg picture, we search for the *stochastic process* that is at the origin of the exponential decay (2.17) of the propagator (2.11).

4.1. Expectation value of the Q particle Hamiltonian

By making use of the explicit expression (3.14) for the momentum operator of the Q particle in the Heisenberg picture, we can easily compute its expectation value in the state

$$|\psi, 0\rangle_N \equiv |\psi\rangle \otimes |0\rangle_N = \int dx\, \psi(x) |x\rangle \otimes |0\rangle_N, \qquad \int_{-\infty}^{+\infty} dx |\psi(x)|^2 = 1 \,. \tag{4.1}$$

The choice of such an uncorrelated initial state is physically consistent: Indeed, we suppose that at initial time $t = 0$ the particle Q is well outside the detector D and moves toward it with constant speed c. It will be clear in the following that the choice of the ground state $|0\rangle_N$ (N spins down) as the initial D state is essential in deriving a stochastic process. For convenience we choose ψ to be symmetrically distributed around the origin and with a compact support. Possible choices for ψ are

$$\psi(x) = (2a)^{-1/2} \theta(a - |x|) e^{ip_0 x/\hbar} \,, \tag{4.2}$$

or a Gaussian wave packet

$$\psi(x) = \left(\frac{1}{2\pi a^2}\right)^{1/4} e^{-x^2/4a^2 + ip_0x/\hbar}, \tag{4.3}$$

truncated for, say, $|x| > a$. Notice again that, owing to (3.3), the wave packet does not disperse. We define

$$\langle c\hat{p}(t)\rangle \equiv {}_N\langle 0,\psi|c\hat{p}(t)|\psi,0\rangle_N \tag{4.4}$$

and, from Eq. (3.14) we obtain

$$\begin{aligned}\langle c\hat{p}(t)\rangle &= {}_N\langle 0,\psi|c\hat{p}(0)|\psi,0\rangle_N + \hbar\omega\, {}_N\langle 0,\psi|\sum_n \sigma_3^{(n)} \sin^2\alpha_n(\hat{x},t)|\psi,0\rangle_N \\ &\quad + i\frac{\hbar\omega}{2}\, {}_N\langle 0,\psi|\sum_n (\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n(\hat{x},t)|\psi,0\rangle_N \\ &\quad + {}_N\langle 0,\psi|\sum_n [V(\hat{x}-x_n) - V(\hat{x}+ct-x_n)](\sigma_+^{(n)}(\hat{x}) + \sigma_-^{(n)}(\hat{x}))|\psi,0\rangle_N .\end{aligned} \tag{4.5}$$

It is easy to see that the last two terms in (4.5) give vanishing contributions. Let us now compute the contributions of the first and the second terms. The evaluation of the first term is straightforward: It is nothing but the initial energy of the Q particle

$$\begin{aligned}{}_N\langle 0,\psi|c\hat{p}(0)|\psi,0\rangle_N &= \langle\psi|c\hat{p}(0)|\psi\rangle = \int dp'\langle\psi|p'\rangle\langle p'|c\hat{p}|\psi\rangle \\ &= \int dp'|\tilde{\psi}(p')|^2\, cp' = cp_0 .\end{aligned} \tag{4.6}$$

On the other hand, the calculation of the second term

$$\begin{aligned}\hbar\omega\, {}_N\langle 0,\psi|\sum_n \sigma_3^{(n)}\sin^2\alpha_n(\hat{x},t)|\psi,0\rangle_N &= -\hbar\omega\langle\psi|\sum_n \sin^2\alpha_n(\hat{x},t)|\psi\rangle \\ &= -\hbar\omega\int dx|\psi(x)|^2\sum_n \sin^2\left[\frac{1}{\hbar}\int_0^t dt'\, V(x+ct'-x_n)\right]\end{aligned} \tag{4.7}$$

is more involved: We can use, without loss of generality (see the appendix), the Fermi-Yang approximation $V(y) = V_0\Omega\delta(y)$, and obtain

$$\begin{aligned}&-\hbar\omega\int dx|\psi(x)|^2\sum_n \sin^2\left[\frac{V_0\Omega}{c\hbar}\int_{x-x_n}^{x+ct-x_n} dy\,\delta(y)\right] \\ &= -\hbar\omega\int dx|\psi(x)|^2\sum_n \sin^2\left[\frac{V_0\Omega}{c\hbar}\,\theta(x+ct-x_n)\theta(x_n-x)\right].\end{aligned} \tag{4.8}$$

If we require

$$a \ll x_1 \quad \text{and} \quad a \ll L = x_N - x_1\,, \tag{4.9}$$

we easily get

$$-\hbar\omega \int dx\,|\psi(x)|^2 \sum_n \sin^2\left[\frac{V_0\Omega}{c\hbar}\,\theta(x+ct-x_n)\right] \tag{4.10}$$

since every spin is located at the far right of the initial wave packet $\psi(x)$, whose finite support is $[-a,a]$, so that the inequality $x(<a) \ll x_n$ holds for the integration variable x. In the weak-coupling, macroscopic limit (2.13), the summation over n can be replaced with an integration as in (2.15) and the above quantity is further reduced to

$$\begin{aligned}
&-\hbar\omega\frac{\bar{n}}{L}\int dx\,|\psi(x)|^2 \int\limits_{x+ct-x_N}^{x+ct-x_1} dz\,\theta(z) \\
&= -\hbar\omega\frac{\bar{n}}{L}\int dx\,|\psi(x)|^2\,[(x+ct-x_1)\theta(x+ct-x_1)\theta(x_N-ct-x) \\
&\quad + (x_N-x_1)\theta(x+ct-x_N)]\,.
\end{aligned} \tag{4.11}$$

Thus, if we restrict our attention to the situation in which Q is still inside D, i.e. $ct < x_N$, we finally obtain

$$\hbar\omega\,{}_N\langle 0,\psi|\sum_n \sigma_3^{(n)}\sin^2\alpha_n(\hat{x},t)|\psi,0\rangle_N \to -\hbar\omega\frac{\bar{n}}{L}(ct-x_1) + \text{b.e.} \tag{4.12}$$

The shorthand notation "b.e." stands for "border effects", namely terms appearing only when $|ct-x_1|,\ |ct-x_N| \leqslant a$, whose explicit expression is easily computed to be

$$-\hbar\omega\frac{\bar{n}}{L}\times\begin{cases}(ct-x_1+a)^2/4a & \text{if } x_1-a\leqslant ct\leqslant x_1+a\,,\\ L-(x_N+a-ct)^2/4a & \text{if } x_N-a\leqslant ct\leqslant x_N+a\,,\end{cases} \tag{4.13}$$

corresponding, respectively, to the situations in which Q is entering D and Q is going out of D.

Having obtained the explicit expressions (4.6) and (4.12) for the first and the second terms in $\langle c\hat{p}(t)\rangle$ [see (4.5)], we reach the following *exact* expression:

$$\langle c\hat{p}(t)\rangle \equiv {}_N\langle 0,\psi|c\hat{p}(t)|\psi,0\rangle_N \to cp_0 - \hbar\omega\frac{\bar{n}}{L}(ct-x_1) + \text{b.e}\,. \tag{4.14}$$

In conclusion, the energy of the Q particle (when Q is *inside* D) decreases linearly with respect to t. It is worth mentioning that what is seen here is an energy-dissipative process: If ω were set equal to zero (as in the original AgBr model [8, 9]), we could not have found such a process.

4.2. Correlation functions of the spin-array Hamiltonian

In order to clarify the stochastic nature of the system, let us calculate the correlation functions of the spin-array Hamiltonian H_D.

Consider the operator $\Delta H_D(t) = H_D(t) - H_D$, which represents the energy stored in the detector between time 0 and t. By using Eqs. (3.12), one obtains

$$\begin{aligned}\Delta H_D(t) &= \sum_n \frac{\hbar\omega}{2}(\sigma_3^{(n)}(t) - \sigma_3^{(n)}) \\ &= -\left[\sum_n \hbar\omega\sigma_3^{(n)} \sin^2\alpha_n(\hat{x}, t) + \sum_n i\frac{\hbar\omega}{2}(\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n(\hat{x}, t)\right].\end{aligned} \tag{4.15}$$

It is easy to see that when the Q particle is inside D, by making use of (4.12), the expectation value of this operator reads

$$\langle \Delta H_D(t)\rangle = {}_N\langle 0, \psi | \Delta H_D(t) | \psi, 0\rangle_N \to \hbar\omega\frac{\bar{n}}{L}(ct - x_1) + \text{b.e.}, \tag{4.16}$$

in agreement with Eq. (4.14): The energy lost by Q is stored in D.

Next we turn our attention to the two-time correlation function, defined by

$$\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle \equiv {}_N\langle 0, \psi | \Delta H_D(t_1)\Delta H_D(t_2) | \psi, 0\rangle_N . \tag{4.17}$$

Its explicit form, by Eq. (4.15), is

$$\begin{aligned}&\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle \\ &= {}_N\langle 0, \psi | \left(\sum_n \hbar\omega\sigma_3^{(n)}\sin^2\alpha_n(\hat{x}, t_1)\right)\left(\sum_m \hbar\omega\sigma_3^{(m)}\sin^2\alpha_m(\hat{x}, t_2)\right) | \psi, 0\rangle_N \\ &\quad + \left(i\frac{\hbar\omega}{2}\right)^2 {}_N\langle 0, \psi | \left[\sum_n (\sigma_+^{(n)}(\hat{x}) - \sigma_-^{(n)}(\hat{x}))\sin 2\alpha_n(\hat{x}, t_1)\right] \\ &\quad \times \left[\sum_m \sigma_+^{(m)}(\hat{x}) - \sigma_-^{(m)}(\hat{x}))\sin 2\alpha_m(\hat{x}, t_2)\right] | \psi, 0\rangle_N + \text{vanishing terms.}\end{aligned} \tag{4.18}$$

The first term yields

$$
\begin{aligned}
&{}_N\langle 0,\psi|\left(\sum_n \hbar\omega\sigma_3^{(n)}\sin^2\alpha_n(\hat{x},t_1)\right)\left(\sum_m \hbar\omega\sigma_3^{(m)}\sin^2\alpha_m(\hat{x},t_2)\right)|\psi,0\rangle_N \\
&= (\hbar\omega)^2\int dx\,|\psi(x)|^2\left[\sum_n \sin^2\left(\frac{1}{\hbar}\int_0^{t_1} dt\, V(x+ct-x_n)\right)\right] \\
&\quad\times\left[\sum_m \sin^2\left(\frac{1}{\hbar}\int_0^{t_2} dt'\, V(x+ct'-x_m)\right)\right],
\end{aligned}
\tag{4.19}
$$

which is written, in the Fermi–Yang approximation $V(x) = V_0\Omega\delta(x)$, as

$$
\begin{aligned}
&(\hbar\omega)^2\int dx\,|\psi(x)|^2\left[\sum_n \sin^2\left(\frac{V_0\Omega}{c\hbar}\theta(x+ct_1-x_n)\theta(x_n-x)\right)\right] \\
&\times\left[\sum_m \sin^2\left(\frac{V_0\Omega}{c\hbar}\theta(x+ct_2-x_m)\theta(x_m-x)\right)\right].
\end{aligned}
\tag{4.20}
$$

The weak-coupling, macroscopic limit (2.13), together with the continuum ansatz (2.15), reduce this quantity to

$$
\begin{aligned}
&\left(\hbar\omega\frac{\bar{n}}{L}\right)^2\int dx\,|\psi(x)|^2[(x+ct_1-x_1)\theta(x+ct_1-x_1)\theta(x_N-ct_1-x) \\
&\quad + L\theta(x+ct_1-x_N)] \\
&\quad\times[(x+ct_2-x_1)\theta(x+ct_2-x_1)\theta(x_N-ct_2-x)+L\theta(x+ct_2-x_N)].
\end{aligned}
\tag{4.21}
$$

Finally, by focusing our attention on the situation in which the Q particle is *inside D*, we obtain

$$
\begin{aligned}
&\left(\hbar\omega\frac{\bar{n}}{L}\right)^2\int dx\,|\psi(x)|^2(x+ct_1-x_1)(x+ct_2-x_1) \\
&= \left(\hbar\omega\frac{\bar{n}}{L}\right)^2(ct_1-x_1)(ct_2-x_1) + \text{b.e.} \quad (\text{for } x_1 < ct_{1,2} < x_N).
\end{aligned}
\tag{4.22}
$$

Let us now calculate the second term in Eq. (4.18), the nonvanishing term of which reads

$$
\left(\frac{\hbar\omega}{2}\right)^2\int dx\,|\psi(x)|^2\sum_n \sin 2\alpha_n(x,t_1)\sin 2\alpha_n(x,t_2). \tag{4.23}
$$

In the usual weak-coupling, macroscopic limit (2.13), we obtain, after a short manipulation,

$$(\hbar\omega)^2 \int dx\,|\psi(x)|^2 \frac{\bar{n}}{L} \int_{x_1}^{x_N} dy\,\theta(x+ct_1-y)\theta(x+ct_2-y)$$

$$= (\hbar\omega)^2 \frac{\bar{n}}{L}\,[c\min(t_1,t_2)-x_1] + \text{b.e.} \quad (\text{for } x_1 < ct_{1,2} < x_N)\,. \tag{4.24}$$

In conclusion, we are led to the following expression for the two-time correlation function:

$$\langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle \to \left(\hbar\omega\frac{\bar{n}}{L}\right)^2 (ct_1-x_1)(ct_2-x_1)$$

$$+ (\hbar\omega)^2 \frac{\bar{n}}{L}[c\min(t_1,t_2)-x_1] + \text{b.e.}\,, \tag{4.25}$$

which is valid when the Q particle is inside D. The border effects will be discussed in the appendix.

4.3. The Wiener process

The results we have obtained so far, (4.16) and (4.25), look quite interesting: Introduce the operator

$$\hat{\Sigma}(t) \equiv \Delta H_D(t) - \langle \Delta H_D(t)\rangle\,, \tag{4.26}$$

where the expectation value is to be evaluated on the state spanned by $|\psi,0\rangle_N \equiv |\psi\rangle \otimes |0\rangle_N$. Then, by Eqs. (4.16) and (4.25), we easily show the following properties

$$\langle \hat{\Sigma}(t)\rangle = 0\,, \tag{4.27}$$

$$\langle \hat{\Sigma}(t_1)\hat{\Sigma}(t_2)\rangle = \langle \Delta H_D(t_1)\Delta H_D(t_2)\rangle - \langle \Delta H_D(t_1)\rangle\langle \Delta H_D(t_2)\rangle$$

$$\to (\hbar\omega)^2 \frac{\bar{n}}{L}|c\min(t_1,t_2)-x_1] + \text{b.e.}, \tag{4.28}$$

valid in the restricted state space spanned by $|\psi,0\rangle_N$. These properties remind us of the characteristics of a *Wiener stochastic process* [17], i.e., a Gaussian process with a variance proportional to $\min(t,t')$, since the second relation (4.28) can be rewritten as

$$\langle \hat{\Sigma}(t_1)\hat{\Sigma}(t_2)\rangle \to (\hbar\omega)^2 \frac{c\bar{n}}{L}\min(\tau_1,\tau_2) + \text{b.e.}, \tag{4.29}$$

in terms of an "interaction time" $\tau_{1,2} \equiv t_{1,2} - x_1/c$. As a matter of fact, we can prove that the operator $\hat{\Sigma}(t)$ really serves as a Wiener process *in the restricted Hilbert space spanned by* $|\psi, 0\rangle_N$, in the weak-coupling, macroscopic limit. To this end, we must show that the process is Gaussian, namely we must prove that the correlation functions of any order can be written as a sum of products of two-time correlation functions over all possible combinations. This will be done in Section 4.4.

4.4. *Characteristic functional*

In order to demonstrate in full generality the Gaussian property of the process, let us consider the characteristic functional

$$\phi[\beta] \equiv \langle e^{\int dt\, \beta(t) \hat{\Sigma}(t)} \rangle \,, \tag{4.30}$$

which is subject to the normalization condition

$$\phi[0] = \langle 1 \rangle = {}_N\langle 0, \psi | \psi, 0 \rangle_N = 1 \,. \tag{4.31}$$

We know that the characteristic functional is the generating functional of correlation functions

$$\langle \hat{\Sigma}(t_1) \hat{\Sigma}(t_2) \cdots \hat{\Sigma}(t_n) \rangle = \left. \frac{\delta^n \phi[\beta]}{\delta\beta(t_1) \cdots \delta\beta(t_n)} \right|_{\beta = 0} \tag{4.32}$$

and that Gaussian processes are characterized by Gaussian characteristic functionals.

By making use of (4.26) together with (4.15) and (4.16), we find

$$\begin{aligned} \phi[\beta] = \int dx |\psi(x)|^2 {}_N\!\left\langle 0 \left| \prod_n e^{-\hbar\omega \int dt\, \beta(t) [\sigma_3^{(n)} \sin^2 \alpha_n(x,t) + \frac{1}{2}(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x)) \sin 2\alpha_n(x,t)]} \right| 0 \right\rangle_N \\ \times e^{-\int dt\, \beta(t) (\Delta H_D(t))} \,. \end{aligned} \tag{4.33}$$

Let us now focus our attention on the factor

$$\begin{aligned} e^{-\hbar\omega \int dt\, \beta(t) [\sigma_3^{(n)} \sin^2 \alpha_n(x,t) + \frac{1}{2}(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x)) \sin 2\alpha_n(x,t)]} &= e^{a_n [b_n \sigma_3^{(n)} + i(\sigma_+^{(n)}(x) - \sigma_-^{(n)}(x))]} \\ &\equiv f(a_n, b_n) \,, \end{aligned} \tag{4.34}$$

where we have introduced the quantities $a_n, b_n \in \Re$:

$$a_n \equiv -\frac{\hbar\omega}{2} \int dt\, \beta(t) \sin 2\alpha_n(x,t) \,, \tag{4.35}$$

$$a_n b_n \equiv -\hbar\omega \int dt\, \beta(t) \sin^2 \alpha_n(x,t) \,. \tag{4.36}$$

We try a disentanglement of f in the following form:

$$f(a,b) = e^{iX(a,b)\sigma_+(x)}\, e^{y(a,b)\sigma_3}\, e^{-iX(a,b)\sigma_-(x)}, \tag{4.37}$$

where, for the moment, the index n has been suppressed for the sake of simplicity and the functions $X(a,b)$, $Y(a,b) \in \Re$ are to be determined later. The determination of X and Y is straightforward but somewhat involved. Differentiation of f, in (4.34), w.r.t. a yields

$$\frac{\partial f(a,b)}{\partial a} = [b\sigma_3 + i(\sigma_+(x) - \sigma_-(x))]f(a,b), \tag{4.38}$$

while the disentangled form of f in (4.37) implies that the same quantity is to be equated with

$$\left[i\frac{\partial X}{\partial a}\sigma_+(x) + \frac{\partial Y}{\partial a} e^{iX\sigma_+(x)}\sigma_3 e^{-iX\sigma_+(x)} - i\frac{\partial X}{\partial a} e^{iX\sigma_+(x)} e^{Y\sigma_3}\sigma_-(x) e^{-Y\sigma_3} e^{-iX\sigma_+(x)} \right] f(a,b). \tag{4.39}$$

The calculation of the terms in square brackets is simple: The second term is calculated to be

$$e^{iX\sigma_+(x)}\sigma_3 e^{-iX\sigma_+(x)} = \sigma_3 - iX[\sigma_3, \sigma_+(x)] = \sigma_3 - 2iX\sigma_+(x), \tag{4.40}$$

while the third term becomes

$$\begin{aligned} e^{iX\sigma_+(x)} e^{Y\sigma_3}\sigma_-(x) e^{-Y\sigma_3} e^{-iX\sigma_+(x)} &= e^{-2Y} e^{iX\sigma_+(x)}\sigma_-(x) e^{-iX\sigma_+(x)} \\ &= e^{-2Y}(\sigma_-(x) + iX\sigma_3 + X^2\sigma_+(x)). \end{aligned} \tag{4.41}$$

Therefore, we have the equality

$$\begin{aligned} & b\sigma_3 + i(\sigma_+(x) - \sigma_-(x)) \\ &\quad = i\frac{\partial X}{\partial a}\sigma_+(x) + \frac{\partial Y}{\partial a}(\sigma_3 - iX2\sigma_+(x)) - i\frac{\partial X}{\partial a} e^{-2Y}(\sigma_-(x) + iX\sigma_3 + X^2\sigma_+(x)), \end{aligned} \tag{4.42}$$

from which we obtain the following set of differential equations:

$$b = \frac{\partial Y}{\partial a} + X\frac{\partial X}{\partial a} e^{-2Y}, \tag{4.43}$$

$$1 = \frac{\partial Y}{\partial a} - 2X\frac{\partial Y}{\partial a} - X^2\frac{\partial X}{\partial a} e^{-2Y}, \tag{4.44}$$

$$1 = \frac{\partial X}{\partial a} e^{-2Y}. \tag{4.45}$$

We can easily solve these equations since they are equivalent to the following two equations:

$$\frac{\partial X}{\partial a} = 1 + 2bX - X^2\,, \tag{4.46}$$

$$\frac{\partial Y}{\partial a} = b - X\,. \tag{4.47}$$

Under the initial condition $X(0,b) = 0$, the solution of (4.46) is readily obtained

$$X(a,b) = \frac{e^{2a\sqrt{b^2+1}} - 1}{(\sqrt{b^2+1}+b) + (\sqrt{b^2+1}-b)e^{2a\sqrt{b^2+1}}}\,. \tag{4.48}$$

Then the function Y is calculated, either from (4.47) or (4.45), to be

$$Y(a,b) = a\sqrt{b^2+1} + \ln\left[\frac{2\sqrt{b^2+1}}{(\sqrt{b^2+1}+b) + (\sqrt{b^2+1}-b)e^{2a\sqrt{b^2+1}}}\right]. \tag{4.49}$$

By plugging the solutions (4.48)–(4.49) into disentanglement formula (4.34) we can evaluate the characteristic functional

$$\begin{aligned}\phi(\beta) &= \int dx\,|\psi(x)|^2\,{}_N\!\left\langle 0\left|\prod_n e^{iX(a_n,b_n)\sigma_+^{(n)}(x)}e^{Y(a_n,b_n)\sigma_3^{(n)}}e^{-iX(a_n,b_n)\sigma_-^{(n)}(x)}\right|0\right\rangle_N \\ &\quad\times e^{-\int dt\,\beta(t)\langle\Delta H_D(t)\rangle} \\ &= \int dx\,|\psi(x)|^2\,e^{-\sum_n Y(a_n,b_n)}e^{-\int dt\,\beta(t)\langle\Delta H_D(t)\rangle}\,.\end{aligned} \tag{4.50}$$

Consider now the weak-coupling, macroscopic limit (2.13), together with (2.15). Obviously $a_n, b_n \sim O(1/\sqrt{N}) \to 0$ and keeping only terms up to order $1/N$ in $Y(a_n,b_n)$, we obtain

$$\begin{aligned}&Y(a_n,b_n) \\ &\quad\to a_n\left(1+\frac{b_n^2}{2}\right) \\ &\qquad + \ln\left[\frac{2(1+b_n^2/2)}{(1+b_n+b_n^2/2)+(1-b_n+b_n^2/2)(1+2a_n(1+b_n^2/2)+(2a_n)^2/2!)}\right] \\ &\quad\simeq -\frac{a_n^2}{2} + a_nb_n\,.\end{aligned} \tag{4.51}$$

Notice that in this limit, the above qualities are expressed as

$$a_n^2 \to \left[\hbar\omega \int dt\, \beta(t)\alpha_n(x,t)\right]\left[\hbar\omega \int dt'\, \beta(t')\alpha_n(x,t')\right], \tag{4.52}$$

$$a_n b_n \to -\hbar\omega \int dt\, \beta(t)\alpha_n^2(x,t). \tag{4.53}$$

Putting these results together and neglecting all border effects, we finally arrive at the explicit expression of the characteristic functional

$$\begin{aligned}\phi[\beta] &\to \int dx\, |\psi(x)|^2\, e^{((\hbar\omega)^2/2)\sum_n \int dt\,dt'\,\beta(t)\alpha_n(x,t)\alpha_n(x,t')\beta(t')}\, e^{\hbar\omega\sum_n \int dt\,\beta(t)\alpha_n^2(x,t)} \\ &\qquad \times e^{-\int dt\,\beta(t)\hbar\omega(\bar{n}/L)(ct-x_1)} \\ &\to e^{(1/2)(\hbar\omega)^2(\bar{n}/L)\int dt\,dt'\,\beta(t)[c\min(t,t')-x_1]\beta(t')}\, e^{\int dt\,\beta(t)\hbar\omega(\bar{n}/L)(ct-x_1)} \\ &\qquad \times e^{-\int dt\,\beta(t)\hbar\omega(\bar{n}/L)(ct-x_1)} \\ &= e^{(1/2)(\hbar\omega)^2(c\bar{n}/L)\int dt\,dt'\,\beta(t)\min(\tau,\tau')\beta(t')}, \end{aligned} \tag{4.54}$$

where the interaction time $\tau = t - x_1/c$ has been introduced as before. The characteristic functional turns out to be Gaussian, which proves that the stochastic process under consideration is Gaussian. We understand from the appearance of $\min(\tau,\tau')$ in the exponent, which represents the variance of a Gaussian process, that this process is nothing but a Wiener process. We stress again that this conclusion is only valid in the restricted state space spanned by $|\psi,0\rangle_N$, in the weak-coupling, macroscopic limit.

5. Comments and outlook

We have analyzed the modified Coleman–Hepp model and brought to light a Wiener process in a restricted portion of the total Hilbert space. The operator $\hat{\Sigma}(t)$ becomes a sort of "noise operator" in the $N \to \infty$ limit (2.13), in the sense of Eqs. (4.27) and (4.28). We also proved the Gaussian white noise properties by starting from the characteristic functional (4.30).

Although the appearance of a stochastic process of some sort could probably be expected on the basis of the stochastic behavior of the propagator (2.17), the emergence of the Gaussian white noise is remarkable, for such a nontrivial Hamiltonian like (2.2). We stressed in [24] that the exponential decay form (2.17) of the propagator is *independent* of ω and therefore the presence of an energy-exchange process is essential for the derivation of the Wiener process, through which the energy of the system is dissipated. We understand that the weak-coupling, macroscopic limit (2.13)–(2.15), that is closely related to Van Hove's limit, plays a crucial role in this respect: It corresponds to a kind of coarse graining and scale-change procedures, some details of which are discussed in the appendix.

It is useful, in this context, to briefly comment on a remark by Leggett [18] that summarizes a widespread opinion among physicists working on these topics. By discussing the role of the environment in connection with the collapse of the wave function, Leggett stressed the central relevance of the problem of dissipation to the quantum measurement theory, and argued that "it is only genuinely dissipative processes, in which the interaction leads to an irreversible exchange of energy between system and environment, which can guarantee that interference is gone beyond the possibility of recovery. Thus, we see that it is not interaction with the environment as such, but specifically dissipation, which is responsible for genuine "decoherence", hence the central relevance of the problem of dissipation to quantum measurement theory". We believe that our analysis contributes to clarify and sharpen the above remark: The behavior just derived, yielding a Wiener process, is certainly related to dephasing ("decoherence") effects of the same kind of those encountered in quantum measurements. The exchange of energy between the particle and the "environment" (our spin system) can be considered practically irreversible. However, the role played by ω is much more subtle, because ω directly contributes to *construct* the stochastic process, as could be seen in Section 4. These remarkable features are manifest in the model here presented and could not be guessed, in our opinion, without an explicit solution. Another important fact, not to be dismissed, is that we have exclusively considered the dynamics within a restricted state space spanned by $|\psi, 0\rangle_N$. We emphasize, once again, that the stochastic process is derived *without* tracing over the states of the macroscopic system D. This is to be contrasted to other work.

We also stress that the link between a dissipative dynamics and a quantum measurement process is not obvious. This point is delicate and somewhat unclear and deserves discussion. In which sense can we talk of "quantum measurement" in the AgBr model? This issue has been analyzed in Refs. [10,11]. One sets up a double-slit experiment, by splitting an incoming Q wave function into two branch waves, only one of which interacts with D. It is possibly to compute exactly several interesting physical quantities, such as the energy "stored" in D after the interaction with Q, as well as the visibility of the interface pattern:

$$\langle H_D \rangle_F = qN\hbar\omega \to \bar{n}\hbar\omega\,, \tag{5.1}$$

$$\mathscr{V} = (1-q)^{N/2} \to e^{-\bar{n}/2} = e^{-\langle H_D \rangle_F / 2\hbar\omega}\,, \tag{5.2}$$

where the sandwich is computed over the final state of the $Q+D$ system after the Q particle has gone through D, q is the "spin-flip" probability (2.6) and the arrow denotes, as usual, the weak-coupling, macroscopic limit $N \to \infty$, $qN = \bar{n} = \text{finite}$.

Clearly, the visibility of the interference pattern decreases as $\bar{n}$ increases, so that interference gradually disappears as the average number $\bar{n}$ of spin flips (dissociated molecules) or, equivalently, the energy exchanged between Q and D increases. However, this is *not enough* to state that we are actually facing a genuine dephasing (decoherence) process, leading to a quantum measurement: Technically, "dephasing" consists in the elimination of the off-diagonal elements of the density matrix of the total $(Q + D)$ system. By contrast, here we are simply observing a dynamical process in which the wave functions of the total system do not completely overlap anymore: In order to shed light on the above point, look at the scalar product between the vectors

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi,0\rangle_N \quad \text{and} \quad |\psi_0(t)\rangle = e^{-iH_0t/\hbar}|\psi,0\rangle_N\,, \tag{5.3}$$

obtained by letting $|\psi,0\rangle_N$, in Eq. (4.1), evolve under the action of the total Hamiltonian H and the free Hamiltonian H_0, respectively. We obtain, in the limit of narrow wave packet [19],

$$\langle\Psi_0(t)|\Psi(t)\rangle = {}_N\langle\psi,0|U_I(t)|\psi,0\rangle_N \to \exp\left(-\bar{n}\frac{(ct-x_1)}{2L}\right), \tag{5.4}$$

where $U_I(t)$ is the same operator defined in (2.4). This result is nothing but Eq. (2.17); the above quantity extracts the net effect of the interaction.

One clearly sees that, *in principle*, the quantum coherence can be recovered: Strictly speaking, the underlying dynamics is unitary and there is no irreversible effect. However, the results obtained in the present paper make us understand that an irreversible process of some sort is present, in the AgBr model. How can we reconcile these two apparently contradictory points of view?

In our opinion, what really provokes the appearance of the Wiener process in the present model is the $N \to \infty$ limit, via Van Hove's diagonal singularity. This practically yields a coarse graining procedure over a certain characteristic time, and discards all effects stemming from fine oscillations over small time periods. In such a limit, the AgBr chain of spins makes a transition from the ordinary unitary representation to a unitary-inequivalent one. Such a phenomenon is characteristic of the many-Hillbert-space theory [14] and yields decoherence effects. Admittedly, we are entering a domain of speculation that should be corroborated by more clear-cut arguments. We feel entitled to put forward the above qualitative comments because the mechanisms at the origin of the stochastic process and the very transition to the unitary inequivalent representation in the present model are not completely clear to us.

We would also like to emphasize that it is not entirely trivial to bring to light the dissipative dynamical processes constituting a quantum measurement: In general, a thermal irreversible process is a probabilistic one, described by master equations, that characterize the approach to thermal equilibrium. On the other hand, in a quantum measurement process, the evolution leads to the so-called collapse of the wave function. The final density matrix, that does not contain off-diagonal terms, depends on the measured observable, on the way one performs the spectral decomposition and on the very measuring apparatus [14,15]. The description of the loss of quantum mechanical coherence in terms of dissipative equations, governing the evolution toward an equilibrium situation of some sort, is therefore a delicate problem, that deserves further investigation.

There are other interesting open problems. For instance, it is well known that the reduced dynamics of a (sub)system in interaction with a larger system (playing the role of reservoir) is well described in terms of quantum dynamical semigroups [25], so that one should be able to derive a master (or Langevin) equal [26] for some dynamical variables of the subsystem (such as the energy-momentum of the Q particle). The derivation of a master or a Langevin equation in the present model would open a door to thoroughly investigate a possible link between a quantum measurement and a genuine dissipative process.

Appendix

In this appendix we shall consider the more realistic situation in which the potential $V(x)$ has a finite width. We shall consider, for simplicity, a square wave packet and potential (actually, the requirement of compact support for ψ and V would suffice)

$$\psi(x) = \frac{1}{\sqrt{2a}}\theta(a-x)\theta(x+a)e^{ip_0x/\hbar}, \tag{A.1}$$

$$V(x) = V_0\theta\left(\frac{\Omega}{2}-x\right)\theta\left(x+\frac{\Omega}{2}\right). \tag{A.2}$$

In this case the expression (2.7) for α_n becomes

$$\begin{aligned}\alpha_n \equiv \alpha_n(x,t) &= \frac{V_0\Omega}{\hbar c}\int\limits_{x-x_n}^{x+ct-x_n}\frac{dy}{\Omega}\theta\left(\frac{\Omega}{2}-y\right)\theta\left(y+\frac{\Omega}{2}\right)\\ &= \frac{\sqrt{q}}{\Omega}\left[\min\left(\frac{\Omega}{2}, x+ct-x_n\right) - \max\left(x-x_n, -\frac{\Omega}{2}\right)\right].\end{aligned} \tag{A.3}$$

The expectation value of the operator $\Delta H_D(t)$ in (4.15), relative to the initial state $|\psi,0\rangle_N$ reads

$$\begin{aligned}\langle\Delta H_D(t)\rangle &= \hbar\omega\int dx\,|\psi(x)|^2\sum_n \sin^2\alpha_n(x,t)\\ &\to \hbar\omega\frac{\bar{n}}{L}\int\limits_{-a}^{a}\frac{dx}{2a}\int\limits_{x_1}^{x_N}\frac{dz}{\Omega^2}\left[\min\left(\frac{\Omega}{2}, x+ct-z\right) - \max\left(x-z, -\frac{\Omega}{2}\right)\right]^2,\end{aligned} \tag{A.4}$$

where, in the last step, we have considered the weak-coupling, macroscopic limit (2.13). We consider now the case $ct > \Omega$ and focus our attention on the situation in which the wave packet is fully inside the potential. This means that

$$x_1 + \frac{\Omega}{2} + a < ct < x_N - \frac{\Omega}{2} - a \,. \tag{A.5}$$

By using (A.5) and since $x_1 \gg \Omega, a$, (A.4) simply becomes

$$\hbar\omega \frac{\bar{n}}{L} \int_{-a}^{a} \frac{dx}{2a} \left[\int_{x_1}^{x+ct-\Omega/2} \frac{dz}{\Omega^2} \Omega^2 + \int_{x+ct-\Omega/2}^{x+ct+\Omega/2} \frac{dz}{\Omega^2} \left(x + ct + \frac{\Omega}{2} - z \right)^2 \right]$$

$$= \hbar\omega \frac{\bar{n}}{L} \int_{-a}^{a} \frac{dx}{2a} \left(x + ct - \frac{\Omega}{6} - x_1 \right) = \hbar\omega \frac{\bar{n}}{L} \left(ct - \frac{\Omega}{6} - x_1 \right). \tag{A.6}$$

Notice that there is no effect due to the wave packet width, since it is considered entirely inside the detector, but the finite width Ω of the potential appears in the above formula, in contrast with (4.16). By following a procedure similar to the previous one, we can compute the second-order correlation function of the operator $\Delta H_D(t)$ and, by making use of the definition (4.26) of $\hat{\Sigma}(t)$ we finally obtain

$$\langle \hat{\Sigma}(t) \rangle = 0 \,,$$

$$\begin{aligned} \langle \hat{\Sigma}(t_1) \hat{\Sigma}(t_2) \rangle \to (\hbar\omega)^2 \frac{\bar{n}}{L} [& \theta(\Delta t - \Omega/c)(ct_1 - x_1) \\ & + \theta(\Delta t)\theta(\Omega/c - \Delta t)\{ct_2 - \Omega - x_1 - h(-\Delta t, \Omega)\} \\ & + \theta(-\Delta t)\theta(\Omega/c + \Delta t)\{ct_1 - \Omega - x_1 - h(-\Delta t, \Omega)\} \\ & + \theta(-\Delta t - \Omega/c)(ct_2 - x_1)] \,, \end{aligned} \tag{A.7}$$

where $\Delta t \equiv t_2 - t_1$ and

$$h(t, \Omega) = \frac{1}{6\Omega^2}(ct + \Omega)((ct + \Omega)^2 - 6\Omega^2) \,. \tag{A.8}$$

From the previous two equations we can observe that, in contrast with (4.27) and (4.28), when the finite width of the potential is taken in account, $\hat{\Sigma}(t)$ is not a Wiener process anymore, unless $\Omega \to 0$ (δ-potential limit). Incidentally, if the time scale is changed like $t = \lambda \bar{t}$ (where t and $\bar{t}$ can be regarded as a microscopic and a macroscopic time respectively) and if we define $\hat{W}(\bar{t}) \equiv \hat{\Sigma}(t)$, then

$$\langle \hat{W}(\bar{t}) \rangle = 0 \tag{A.9}$$

$$\begin{aligned}\langle \hat{W}(\bar{t}_1)\hat{W}(\bar{t}_2)\rangle \to (\hbar\omega)^2\frac{\bar{n}}{\bar{L}}[&\theta(\Delta\bar{t}-\Omega/\lambda c)(c\bar{t}_1-\bar{x}_1)\\ &+\theta(\Delta\bar{t})\theta(\Omega/\lambda c-\Delta\bar{t})\{c\bar{t}_2-\Omega/\lambda-\bar{x}_1-h(-\Delta\bar{t},\Omega/\lambda)\}\\ &+\theta(-\Delta\bar{t})\theta(\Delta\bar{t}+\Omega/\lambda c)\{c\bar{t}_1-\Omega/\lambda-\bar{x}_1-h(\Delta\bar{t},\Omega/\lambda)\}\\ &+\theta(-\Delta\bar{t}-\Omega/\lambda c)(c\bar{t}_2-\bar{x}_1)]\,,\end{aligned} \tag{A.10}$$

where $\bar{x}_1 = x_1/\lambda$ and $\bar{L} = L/\lambda$. In this case, only when $|\bar{t}_2 - \bar{t}_1| \gg \Omega/\lambda c$ (or $\lambda \to \infty$ with $\bar{x}_1, \bar{L}, \bar{t} < \infty$, which is equivalent to a time scale transformation), we reobtain a proper Wiener process. In other words, the δ-potential limit can be *regarded as a realization of the macroscopic time-scale transformation.*

The above considerations bring to light the close link with Van Hove's $\lambda^2 T$ limit, as discussed in [11]. This can be easily evinced by observing that q, in Eq. (2.13), is nothing but the square of a coupling constant (Van Hove's λ), and that $N(\propto L)$ can be considered proportional to the total interaction time T. Notice also that the "lattice spacing" d, the inverse of which corresponds to a density in our one-dimensional model, can be kept finite in the limit. Obviously, in such a case, we have to express everything in terms of scaled variables such as $\bar{t} = t/\lambda$, $\bar{x} = x/\lambda$ and $\zeta \equiv a/L$, where a is the size of the wave packet.

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