Taylor & Francis
Taylor & Francis Group

On noise-induced superselection rules

S. PASCAZIO†‡

†Dipartimento di Fisica, Università di Bari, I-70126 Bari, Italy
‡Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126
Bari, Italy

Abstract. The dynamical properties of a quantum system can be profoundly influenced by its environment. Usually, the environment provokes decoherence and its action on the system can often be schematized by adding a noise term in the Hamiltonian. However, other scenarios are possible: we show that by increasing the strength of the noise, the Hilbert space of the system gradually splits into invariant subspaces, among which transitions become increasingly difficult. The phenomenon is equivalent to the formation of the quantum Zeno subspaces. We explore the possibility that noise can *prevent*, rather than provoke decoherence.

1. Introduction

Interactions with the environment provoke decoherence [1] on quantum systems. The physical mechanisms at the origin of the loss of quantum coherence are diverse and can be heuristically modelled in many different ways. However, usually, these mechanisms can be viewed as yielding a 'disturbance' or a phase randomization of some sort. For this reason, it is often allowable to neglect the detailed features of the environment and schematize its global effect on the system by means of noise terms in the Hamiltonian of the latter. One often reads that noise provokes decoherence. There are, however, noteworthy exceptions: a large noise can help to stabilize a quantum system, suppressing transitions to other states. This mechanism was understood in the late 1970s [2] and enabled one to explain the stability of certain chiral molecules. It is therefore worth investigating in which sense noise can yield superselection rules and whether/when noise can *prevent*, rather than provoke decoherence.

Several strategies have been proposed during the last few years in order to counter decoherence, in particular in the context of quantum computation [3]. Quantum error correcting codes [4], decoherence-free subspaces [5], 'bang-bang' pulses and dynamical decoupling [6] are just some examples. Other interesting proposals make use of the quantum Zeno effect (QZE) [7]) and the recently introduced quantum Zeno subspaces [8]. Moreover, the possibility of preserving quantum coherence by means of a stochastic control has been recently advocated by Mancini *et al.* [9], who also emphasized the links with the quantum Zeno effect [10]. The unification of these schemes under the same basic ideas [11] enables one to look at this problem from a broader perspective.

In this paper we shall look in detail at the afore-mentioned noise-based strategy to inhibit transitions (and therefore, perhaps, to control decoherence). We shall start by looking at a simple example studied by Blanchard *et al.* [12] and Berry [13]. We first reinterpret some of their findings in terms of the QZE [14] and then broaden the applicability of the method to include a wider class of quantum Zeno phenomena.

2.   The model

The model studied by Blanchard *et al.* [12] describes a two-level system interacting with an environment according to the Hamiltonian

$$H = \alpha\sigma_1 + \beta\eta(t)\sigma_3, \tag{1}$$

where α and β are real constants and σ_i ($i = 1, 2, 3$) are Pauli matrices. The action of the environment on the system is modelled by the stochastic term $\eta\sigma_3$, where

$$\langle\eta(t)\rangle = 0, \quad \langle\eta(t)\eta(t')\rangle = \delta(t - t'), \tag{2}$$

the brackets denoting the average over all possible realizations of the white noise η. In terms of the Wiener process

$$\mathrm{d}W(t) \equiv W(t + \mathrm{d}t) - W(t) = \int_t^{t+\mathrm{d}t} \eta(s)\,\mathrm{d}s,$$

$$\langle\mathrm{d}W(t)\rangle = 0, \quad \langle\mathrm{d}W(t)\,\mathrm{d}W(t)\rangle = \mathrm{d}t, \tag{3}$$

the Ito–Schrödinger equation reads ($\hbar = 1$)

$$|\mathrm{d}\psi\rangle = -\mathrm{i}\alpha\sigma_1|\psi\rangle\,\mathrm{d}t - \mathrm{i}\beta\sigma_3|\psi\rangle \circ \mathrm{d}W = \left(-\mathrm{i}\alpha\sigma_1 - \frac{1}{2}\beta^2\right)|\psi\rangle\,\mathrm{d}t - \mathrm{i}\beta\sigma_3|\psi\rangle\mathrm{d}W, \tag{4}$$

where $\circ$ denotes the Stratonovich product and $|\psi\rangle = (|\psi_+\rangle, |\psi_-\rangle)^{\mathrm{T}}$ is a two-component spinor (we work in the basis of the eigenstates of σ_3). When $\beta = 0$, the above equation yields coherent (Rabi) oscillations between the two eigenstates of σ_3. This Hamiltonian schematizes a two-level system interacting with an environment, whose action is 'summarized' by means of a white noise multiplying an operator of the system. The model describes a superconducting ring enclosing a quantized magnetic flux. Coherent tunnelling between the two flux configurations is possible if the system is very well isolated from its environment ($\beta = 0$). In general, coherence is gradually lost when $\beta \neq 0$; however, as we shall see, it is of primary importance to focus on the time scales of the decoherence process.

The polarization (Bloch) vector

$$\mathbf{x}(t) = \langle\psi|\boldsymbol{\sigma}|\psi\rangle, \tag{5}$$

satisfies the stochastic differential equation

$$\mathrm{d}\mathbf{x}(t) = A\mathbf{x}(t)\,\mathrm{d}t + B\mathbf{x}(t)\,\mathrm{d}W(t), \tag{6}$$

where

$$A = \begin{pmatrix} -2\beta^2 & 0 & 0 \\ 0 & -2\beta^2 & -2\alpha \\ 0 & 2\alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2\beta & 0 \\ 2\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The Bloch vector is therefore a stochastic process, whose third component $z = \langle\psi_+|\psi_+\rangle - \langle\psi_-|\psi_-\rangle$ yields information on the probability of finding the system

in one of the eigenstates of σ_3. The density matrix of a two-level system (like the one considered above) can always be expressed in terms of the Bloch vector (5), according to the formula

$$\rho = \frac{1}{2}(\mathbf{1} + \mathbf{x} \cdot \boldsymbol{\sigma}), \tag{8}$$

where $\mathrm{Tr}\,(\rho) = 1$ (normalization) and $\mathrm{Tr}\,(\rho\boldsymbol{\sigma}) = \mathbf{x}$. Pure states are characterized by $\|\mathbf{x}\| = 1$ and it is easy to check that (5) yields

$$\|\mathbf{x}(t)\|^2 \equiv x^2(t) + y^2(t) + z^2(t) = 1, \quad \forall t: \tag{9}$$

the state remains pure for every individual realization of the stochastic process. If the average (2)–(3) (denoted with a bar throughout) is computed, one gets a Gorini–Kossakowski–Sudarshan–Lindblad equation [15]

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} = -\mathrm{i}[\alpha\sigma_1, \overline{\rho}] - \beta^2(\overline{\rho} - \sigma_3\overline{\rho}\sigma_3). \tag{10}$$

By making use of the explicit expression (8) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{x} = -2\beta^2\overline{x}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\overline{y} = -2\alpha\overline{z} - 2\beta^2\overline{y}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\overline{z} = 2\alpha\overline{y}, \tag{11}$$

whose solution is

$$\begin{aligned}
\overline{x}(t) &= \overline{x}(0)\exp(-2\beta^2 t), \\
\overline{y}(t) &= \exp(-\beta^2 t)(\overline{y}(0)\cos\omega t + c_1\sin\omega t), \\
\overline{z}(t) &= \exp(-\beta^2 t)(\overline{z}(0)\cos\omega t + c_2\sin\omega t),
\end{aligned} \tag{12}$$

where $c_1 = (-\beta^2\overline{y}(0) - 2\alpha\overline{z}(0))/\omega$, $c_2 = (\beta^2\overline{z}(0) + 2\alpha\overline{y}(0))/\omega$ and $\omega = (4\alpha^2 - \beta^4)^{1/2}$. Note that if $4\alpha^2 - \beta^4 < 0$, ω becomes purely imaginary and the solution is simply obtained by replacing the trigonometric functions in (12) with the hyperbolic ones: $\cos\omega t \to \cosh\omega t$, $\sin\omega t \to \sinh\omega t$.

3. Large noise versus quantum Zeno effect

Different dynamical regimes can be obtained by varying the coupling β with the environment: if β is small, the interaction with the environment is weak and the system undergoes coherent quantum oscillations between its two states. If, on the other hand, β is large, these oscillations are hindered and the system becomes 'localized' in one of its two states [12, 13].

Let us clarify the links between this localization phenomenon and the quantum Zeno effect [14]. Prepare the system in the initial state $\overline{x}(0) = \overline{y}(0) = 0, \overline{z}(0) = 1$ (all particles in state $|\psi_+\rangle$). If the coupling with the environment is large $\beta^2 \gg 2\alpha$, the solution is

$$\begin{aligned}
\overline{\mathbf{x}}(t) &= \exp(-\beta^2 t)\begin{pmatrix} 0 \\ -(2\alpha/\omega)\sinh\omega t \\ \cosh\omega t + (\beta^2/\omega)\sinh\omega t \end{pmatrix} \\
&\xrightarrow{\text{large }\beta^2} \begin{pmatrix} 0 \\ 0 \\ \exp[-(2\alpha^2/\beta^2)t] \end{pmatrix} \xrightarrow{\beta\to\infty} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\end{aligned} \tag{13}$$

where 'large β^2' means $\beta^2 \gg 2\alpha, t^{-1}$ and we neglected terms $O(\alpha/\beta^2)$ in the third expression. As one can see, when β is large, the oscillations are hindered and the system tends to remain in its initial state. Notice also that in the above formulas one implicitly assumes that $t < \infty$. This 'halting' of the quantum evolution due to strong coupling with the environment is familiar in a variety of physical situations [2].

Let us now take a different approach. Assume that the system is not coupled to the environment, $\beta = 0$, but *frequent* measurements are performed on the system in order to ascertain whether it is localized in one of the eigenstates of σ_3 ($|\psi_+\rangle$ or $|\psi_-\rangle$). This is the usual framework of 'pulsed' observation, typical of the quantum Zeno effect. The solution of the Bloch equation is (no average is actually needed, but we keep the bar for ease of comparison with the previous case)

$$\bar{\mathbf{x}}(t) = \begin{pmatrix} 0 \\ -\sin 2\alpha t \\ \cos 2\alpha t \end{pmatrix} \overset{\text{small } t}{\simeq} \begin{pmatrix} 0 \\ -2\alpha t \\ 1 - 2\alpha^2 t^2 \end{pmatrix}, \tag{14}$$

where 'small t' means $t \ll 2\alpha = \omega^{-1}$. It is easy to check [14] that if N σ_3-measurements are performed at time intervals δt one obtains

$$\bar{\mathbf{x}}(t) = \begin{pmatrix} 0 \\ 0 \\ [1 - 2\alpha^2(t/N)^2]^N \end{pmatrix} \overset{\text{large } N}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \\ \exp[-(2\alpha^2 \delta t)t] \end{pmatrix} \overset{\delta t \to 0}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{15}$$

Notice that we are implicitly assuming that $t < \infty$. Once again, the oscillations are hindered.

The two situations analysed in this section, large coupling with the environment and frequent measurements, yield the same physical effect. The two regimes can also be *quantitatively* compared: if

$$\beta^{-2} = \delta t, \tag{16}$$

(13) and (15) are asymptotically identical. A (σ_3) white noise of large strength β and a series of frequent (σ_3) observations at short time intervals δt slow down (and eventually halt) the evolution of an eigenstate of σ_3 (initial condition $\bar{z}(0) = 1$).

4.   The general framework

We can now generalize the results of the previous sections in order to try and understand the reasons for the occurrence of the 'localization' phenomenon in the initial state (which was also an eigenstate of σ_3). Since a large noise is physically equivalent to the quantum Zeno effect and since the latter is physically equivalent to dynamical decoupling [11] and leads to the formation of the quantum Zeno subspaces [8], one expects that the 'localization' observed in the preceding sections can be viewed as a *dynamical* phenomenon, due to the formation of a Zeno subspace. This expectation is correct and can be put on firm ground.

Let a quantum system be described by the time-dependent Hamiltonian

$$H_K = H_0 + \eta(t)KH_1, \tag{17}$$

where H_0 and H_1 are Hermitian, time-independent operators. The action of the environment on the system is schematized by the stochastic term ηKH_1, where η is

a white noise and K is the coupling constant. The Hamiltonian (1) is a particular case of the above.

The evolution is

$$|d\psi\rangle = -iH_0|\psi\rangle \, dt - iKH_1|\psi\rangle \circ dW = \left(-iH_0 - \frac{1}{2}K^2H_1^2\right)|\psi\rangle \, dt - iKH_1|\psi\rangle \, dW, \tag{18}$$

or alternatively

$$\frac{d}{dt}\overline{\rho} = -i[H_0, \overline{\rho}] - \frac{K^2}{2}\{H_1^2, \overline{\rho}\} + K^2 H_1 \overline{\rho} H_1 = (\mathcal{L}_0 + K^2\mathcal{L})\overline{\rho}, \tag{19}$$

where $[\cdot, \cdot]$ is the commutator, $\{\cdot, \cdot\}$ is the anticommutator and $\mathcal{L}_0$ and $\mathcal{L}$ are the free and dissipative part of the Liouvillian, respectively.

Let us endeavour to understand what happens when K becomes large. Consider the limiting evolution operator in the interaction picture

$$\mathcal{U}(t) = \lim_{K\to\infty} U_K^I(t) = \lim_{K\to\infty} U_1^\dagger(t) U_K(t), \tag{20}$$

where

$$U_K(t) = \exp(-iH_K t),$$
$$U_1(t) = \exp\left(-iKH_1\int_0^t \eta(t')\,dt'\right) = \exp(-iKH_1 W(t)), \tag{21}$$

all evolution operators acting *à la* Ito on the wave function. U_K^I satisfies the Schrödinger equation in the interaction picture

$$i\partial_t U_K^I(t) = H_0^I(t) U_K^I(t), \quad H_0^I(t) = U_1(t)^\dagger H_0 U_1(t) \tag{22}$$

and it is not difficult to show, by adapting the proof of [8], that in the large-K limit the evolution operator becomes diagonal with respect to H_1:

$$[\mathcal{U}(t), P_n] = 0, \quad \text{where} \quad H_1 P_n = \eta_n P_n, \tag{23}$$

P_n being the orthogonal projection onto $\mathcal{H}_{P_n}$, the eigenspace of H_1 belonging to the eigenvalue η_n. (Note that in equation (23) the eigenvalues are in general distinct, $\eta_n \neq \eta_m$ for $n \neq m$, and the $\mathcal{H}_{P_n}$s are in general multidimensional.) Moreover, the limiting evolution operator has the explicit form

$$\mathcal{U}(t) = \exp(-iH_{\text{diag}}t), \quad H_{\text{diag}} = \sum_n P_n H_0 P_n \equiv \hat{P}H_0. \tag{24}$$

In other words, in the $K \to \infty$ limit an effective superselection rule arises and the total Hilbert space is split into (Zeno) subspaces $\mathcal{H}_{P_n}$ that are invariant under the evolution. The dynamics within each Zeno subspace $\mathcal{H}_{P_n}$ is governed by the diagonal part $P_n H_0 P_n$ of the free Hamiltonian H_0. We stress that the superselection rules discussed here are a consequence of the Zeno dynamics (strong coupling) and are equivalent to the celebrated 'W³' ones [16].

We also notice that the very same Zeno subspaces could be obtained by looking for the eigenspace of the dissipative part of the Liouvillian $\mathcal{L}$ in (19) corresponding to the null eigenvalue:

$$\mathcal{L}\hat{P} = 0, \tag{25}$$

where $\hat{P}$ is defined in (24). Since a vanishing eigenvalue implies in this case no dissipation, the corresponding Zeno subspaces can be viewed as decoherence-free.

5.   Comments

We have analysed a method to inhibit quantum transitions that makes use of a large noise. The method has been well known for a long time [2, 12, 13], but the interpretation in terms of the quantum Zeno subspaces [8] is novel. A complete theory, valid for general Gorini–Kossakowski–Sudarshan–Lindblad equations [15] will be presented elsewhere, as it is far from being trivial. Such a complete theory would be required, in particular, in order to fully understand some recent proposals [9, 10] that focus on the preservation of quantum coherence by stochastic control. The real problem, when one endeavours to control decoherence [17] is the occurrence of the *inverse* Zeno effect [18] and the key role played by the form factors of the interaction. In order to take the consequences of the inverse Zeno effect into account it is important to accurately model the interaction between the quantum system and its environment. It is well known that there is no general recipe in order to obtain 'noise' terms from the total Hamiltonian (describing the environment + the system) in a rigorous way. As a matter of fact, this programme can be carried out only in some particular cases [19], which have played a fundamental role in clarifying the features of quantum dissipative phenomena [20]. However, strong coupling regimes should be handled separately and the validity of the interaction Hamiltonian in (17), when one endeavours to model the physical system of interest, must be carefully pondered over. Other issues that are certainly worth exploring, in this context, are the links with the so-called continuous measurements [21] and the mechanisms yielding stochastic resonance [22].

References

[1] GIULINI, D., JOOS, E., KIEFER, C., KUPSCH, J., STAMATESCU, I.-O., and ZEH, H.-D., 1996, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Berlin: Springer); NAMIKI, M., PASCAZIO, S., and NAKAZATO, H., 1997, *Decoherence and Quantum Measurements* (Singapore: World Scientific); ZUREK, W. H., 2003, *Rev. mod. Phys.*, **74**, 715.
[2] SIMONIUS, M., 1978, *Phys. Rev. Lett.*, **40**, 980; HARRIS, R. A., and STODOLSKY, L., 1978, *Phys. Lett.* B, **78**, 313; 1981, *J. chem. Phys.*, **74**, 2145; 1982, *Phys. Lett.* B, **116**, 464; CINA, J. A., and HARRIS, R. A., 1995, *Science*, **267**, 832.
[3] GALINDO, A., and MARTIN-DELGADO M. A., 2002, *Rev. mod. Phys.*, **74**, 347; BOUWMEESTER, D., EKERT, A., and ZEILINGER, A. (eds), 2000, *The Physics of Quantum Information* (Berlin: Springer); NIELSEN, M. A., and CHUANG, I. L., 2000, *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press).
[4] SHOR, P. W., 1995, *Phys. Rev.* A, **52**, 2493; CALDERBANK, A. R., and SHOR, P. W., 1996, *Phys. Rev.* A, **54**, 1098; STEANE, A., 1996, *Proc. R. Soc.* (London) A, **452**, 2551; *Phys. Rev. Lett.*, **77**, 793; PRESKILL, J., 1999, *Introduction to Quantum Computation*

and Information, edited by H. K. Lo, S. Popescu and T. P. Spiller (Singapore: World Scientific).

[5] PALMA, G. M., SUOMINEN, K. A., and EKERT, A. K., 1996, *Proc. R. Soc.* (London) A, **452**, 567; DUAN, L. M., and GUO, G. C., 1997, *Phys. Rev. Lett.*, **79**, 1953; ZANARDI, P., and RASETTI, M., 1997, *Phys. Rev. Lett.*, **79**, 3306; LIDAR, D. A., CHUANG, I. L., and WHALEY, K. B., 1998, *Phys. Rev. Lett.*, 1998, **81**, 2594; KNILL, E., LAFLAMME, R., and VIOLA, L., 2000, *Phys. Rev. Lett.*, **84**, 2525.

[6] VIOLA, L., and LLOYD, S., 1998, *Phys. Rev.* A, **58**, 2733; VIOLA, L., KNILL, E., and LLOYD, S., 1999, *Phys. Rev. Lett.*, **82**, 2417; 1999, *ibid.*, **83**, 4888; 2000, *ibid.*, **85**, 3520; ZANARDI, P., 1999, *Phys. Lett.* A, **258**, 77; VITALI, D., and TOMBESI, P., 1999, *Phys. Rev.* A, **59**, 4178; 2001, *ibid.*, **65**, 012305; UCHIYAMA, C., and AIHARA, M., 2002, *Phys. Rev.* A, **66**, 032313; BYRD, M. S., and LIDAR, D. A., 2002, *Quantum Informat. Process.*, **1**, 19; 2003, *Phys. Rev.* A, **67**, 012324.

[7] VON NEUMANN, J., 1955, *Mathematical Foundation of Quantum Mechanics* (Princeton: Princeton University Press); BESKOW, A., and NILSSON, J., 1967, *Arkiv Fys.*, **34**, 561; KHALFIN, L. A., 1968, *JETP Lett.*, **8**, 65; MISRA, B., and SUDARSHAN, E. C. G., 1977, *J. math. Phys.*, **18**, 756; FACCHI, P., and PASCAZIO, S., 2001, *Progress in Optics*, Vol. 42, edited by E. Wolf (Amsterdam: Elsevier), chapter 3, p. 147.

[8] FACCHI, P., and PASCAZIO, S., 2002, *Phys. Rev. Lett.*, **89** 080401; 2003, *Proceedings of the XXII Solvay Conference on Physics*, edited by I. Antoniou, V. A. Sadovnichy and H. Walther (Singapore: World Scientific), p. 251 [quant-ph/0207030].

[9] MANCINI, S., VITALI, D., TOMBESI, P., and BONIFACIO, R., 2002, *Europhys. Lett.*, **60**, 498; 2002, *J. Opt.* B, **4**, S300.

[10] MANCINI, S., and BONIFACIO, R., 2001, *Phys. Rev.* A, **64**, 042111.

[11] FACCHI, P., LIDAR, D. A., and PASCAZIO, S., 2004, *Phys. Rev.* A (in the press) [quant-ph/0303132].

[12] BLANCHARD, PH., BOLZ, G., CINI, M., DE ANGELIS, G. F., and SERVA, M., 1994, *J. stat. Phys.*, **75**, 749.

[13] BERRY, M., 1995, In: *Fundamental Problems in Quantum Theory*, edited by D. M. Greenberger and A. Zeilinger (*Ann. New York Ac. Sci.*, **755**, 303).

[14] NAKAZATO, H., and PASCAZIO, S., 1999, *J. Superconductivity*, **12**, 843.

[15] GORINI, V., KOSSAKOWSKI, A., and SUDARSHAN, E. C. G., 1976, *J. math. Phys.*, **17**, 821; LINDBLAD, G., 1976, *Comm. math. Phys.*, **48**, 119; see also ALICKI, R., and LENDI, K., 1987, *Quantum Dynamical Semigroups and Applications*, *Lecture Notes in Physics*, Vol. 286 (Berlin: Springer); HOLEVO, A. S., 1989, *Proc. Steklov Inst. Math.*, **191**, 131.

[16] WICK, G. C., WIGHTMAN, A. S., and WIGNER, E. P., 1952, *Phys. Rev.*, **88**, 101; 1970, *Phys. Rev.* D, **1**, 3267.

[17] CALARCO, T., DATTA, A., FEDICHEV, P., PAZY, E., and ZOLLER, P., 2003, *Phys. Rev.* A, **68**, 012310; TASAKI, S., TOKUSE, A., FACCHI, P., and PASCAZIO, S., 2002, quant-ph/0210129.

[18] LANE, A. M., 1983, *Phys. Lett.* A, **99**, 359; SCHIEVE, W. C., HORWITZ, L. P., and LEVITAN, J., 1989, *Phys. Lett.* A, **136**, 264; KAULAKYS, B., and GONTIS, V., 1997, *Phys. Rev.* A, **56**, 1131; THUN, K., and PEŘINA, J., 1998, *Phys. Lett.* A, **249**, 363; FACCHI, P., and PASCAZIO, S., 2000, *Phys. Rev.* A, **62**, 023804; ELATTARI, B., and GURVITZ, S. A., 2000, *Phys. Rev.* A, **62**, 032102; KOFMAN, A. G., and KURIZKI, G., 2000, *Nature*, **405**, 546; FACCHI, P., NAKAZATO, H., and PASCAZIO, S., 2001, *Phys. Rev. Lett.*, **86**, 2699; KOSHINO, K., and SHIMIZU, A., 2003, *Phys. Rev.* A, **67**, 042101.

[19] FORD, G. W., KAC, M., and MAZUR, P., 1965, *J. math. Phys.*, **9**, 504; MAGALINSKIJ, V. B., 1959, *Zh. Eksp. Teor. Fiz.*, **36**, 1942 [1959, *Sov. Phys. JEPT*, **9**, 1381]; FEYNMAN, R. P., and VERNON, F. L., 1963, *Ann. Phys.* (New York), **24**, 118; ULLERSMA, P., 1966, *Physica*, **32**, 27; ZWANZIG, R., 1973, *J. stat. Phys.*, **9**, 215; CALDEIRA, A. O., and LEGGETT, A. J., 1981, *Phys. Rev. Lett.*, **46**, 211; 1983, *Physica* A, **121**, 587; 1985, *Phys. Rev.* A, **31**, 1059; HAKIM, V., and AMBEGAOKAR, V., 1985, *Phys. Rev.* A, **32**, 423; FORD, G. W., and KAC, M., 1987, *J. stat. Phys.*, **46**, 803; GRABERT, H., SCHRAMM, P., and INGOLD, G. L., 1987, *Phys. Rev. Lett.*, **58**, 58; FORD, G. W., LEWIS, J. T., and O'CONNELL, R. F., 1988, *Phys. Rev.* A, **37**, 4419.

[20] Wax, N. (ed.), 1954, *Selected Papers on Noise and Stochastic Processes* (New York: Dover); van Kampen, N. G., 1981, *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland); Gardiner, C. W., 1990, *Handbook of Stochastic Methods* (Berlin: Springer-Verlag); Gardiner, C. W., and Zoller, P., 2000, *Quantum Noise* (Berlin: Springer).

[21] Peres, A., 1980, *Am. J. Phys.*, **48**, 931; Venugopalan, A., and Ghosh, R., 1995, *Phys. Lett.* A, **204**, 11; Plenio, M. P., Knight, P. L., and Thompson, R. C., 1996, *Optics Commun.*, **123**, 278; Berry, M. V., and Klein, S., 1996, *J. mod. Optics*, **43**, 165; Mihokova, E., Pascazio, S., and Schulman, L. S., 1997, *Phys. Rev. A*, **56**, 25; Schulman, L. S., 1998, *Phys. Rev. A*, **57**, 1509; Luis, A., and Sánchez-Soto, L. L., 1998, *Phys. Rev. A*, **57**, 781; Panov, A. D., 1999, *Phys. Lett.* A, **260**, 441; Řeháček, J., Peřina, J., Facchi, P., Pascazio, S., and Mišta, L., 2000, *Phys. Rev. A*, **62**, 013804; Facchi, P., and Pascazio, S., 2000, *Phys. Rev. A*, **62**, 023804; Militello, B., Messina, A., and Napoli, A., 2001, *Phys. Lett.* A, **286**, 369; Luis, A., 2001, *Phys. Rev. A*, **64**, 032104.

[22] Benzi, R., Sutera, A., and Vulpiani, A., 1981, *J. Phys. A*, **14**, L453; Benzi, R., Sutera, A., Parisi, G., and Vulpiani, A., 1983, *SIAM (Soc. Ind. appl. Math.) J. appl. Math.*, **43**, 565; Gammaitoni, L., Hänggi, P., Jung, P., and Marchesoni, F., 1998, *Rev. mod. Phys.*, **70**, 223; Wellens, T., and Buchleitner, A., 2000, *Phys. Rev. Lett.*, **84**, 5118.