Long-time memory in non-Markovian evolutions

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If the dynamics of an open quantum system is non-Markovian, its asymptotic state strongly depends on the initial conditions, even if the dynamics possesses an invariant state. This is the very essence of memory effects. In particular, the asymptotic state can remember and partially preserve its initial entanglement. Interestingly, even if the non-Markovian evolution relaxes to an equilibrium state, this state needs not be invariant. Therefore, the noninvariance of equilibrium becomes a clear sign of non-Markovianity.

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I. INTRODUCTION

Open quantum systems and their dynamical features are attracting increasing attention nowadays. Reasons for their interest are twofold. On one hand, they are of paramount importance in the study of the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [1,2]. On the other hand, the robustness of quantum coherence and entanglement against the detrimental effects of the environment is one of the major focuses in quantum-enhanced applications, as both entanglement and quantum coherence are basic resources in modern quantum technologies, such as quantum communication, cryptography, and computation [3].

The detailed characteristics of the dynamical evolution are far from being obvious and are often quite surprising. For example, while the coherence of single qubits in Markovian environments decays exponentially, the evolution of the entanglement between two qubits markedly differs and may completely disappear at a *finite* time (and eventually revive later) [4], a phenomenon known as "entanglement sudden death," that has been recently experimentally demonstrated [5] and analyzed from different perspectives [6].

In this article we will focus on *non-Markovian* evolutions and will show that they define a completely new kind of quantum dynamics. In particular, this leads to the modification of the characteristic exponential relaxation law known from Markovian evolutions. Interestingly, we will show that even if the non-Markovian evolution relaxes to an equilibrium state this state need not be invariant. This can never happen in the Markovian case. Therefore, the noninvariance of equilibrium becomes a clear sign of non-Markovianity. It turns out that the asymptotic state of the system depends on the initial conditions, even if the non-Markovian dynamics possesses an invariant state. For composed systems this implies that the asymptotic states can remember (and partially preserve) its initial entanglement. These conclusions will be illustrated by several examples and pave the way toward a more general comprehension and practical exploitation of non-Markovian evolutions.

II. PRELIMINARY IDEAS

A. Non-Markovian dynamics

The usual approach to the dynamics of an open quantum system consists of applying the Markovian approximation, which leads to the local master equation

$$\dot{\rho}_t = L\rho_t, \tag{1}$$

where ρ_t is the density matrix of the system investigated and L the time-independent generator of the dynamical semigroup. This can be formally solved,

$$\rho_t = e^{tL}\rho = \Lambda_t\rho \quad (t \geqslant 0,\ \rho = \rho_{t=0}), \tag{2}$$

and it is well known that under certain conditions on L [7], the dynamics Λ_t is completely positive (CP) and trace preserving [2,8].

Let us study the behavior of quantum coherence under non-Markovian evolutions. For the sake of simplicity, we shall restrict our attention to finite-level systems. A popular non-Markovian generalization of (1) is the nonlocal equation

$$\dot{\rho}_t = \int_0^t L_{t-\tau}\,\rho_\tau\,d\tau, \tag{3}$$

in which quantum memory effects are taken into account through the introduction of the memory kernel L_τ; this simply means that the rate of change of the state ρ_t at time t depends on its history (starting at $t = 0$). The Markovian master Eq. (1) is reobtained when $L_\tau = 2\delta(\tau)L$. The time-dependent kernel L_τ is usually referred to as the generator of the non-Markovian master equation. Equation (3) applies to a variety of situations, for example, when the particle is born in the medium in which it propagates (neutrinos in a stellar medium [9] or pairs of neutral kaons in the gravitation field of a laboratory [10]).

One of the fundamental problems in the theory of non-Markovian master equations is finding those conditions on L_τ that ensure that the time evolution resulting from (3),

$$\rho \longrightarrow \rho_t = \Lambda_t\rho, \tag{4}$$

is CP and trace preserving [11–18]. Let us observe that this problem may be reformulated as follows [19]: any CP solution

Λ_t of Eq. (3) may be represented by

$$\Lambda_t = \mathbb{1} + \int_0^t \Phi_\tau \, d\tau, \tag{5}$$

where the maps Φ_τ satisfy $\mathrm{Tr}\, \Phi_\tau \rho = 0$ for all ρ. This condition guaranties that Λ_t is trace preserving. It is easy to show that the Laplace transform of the generator L_τ of the non-Markovian master equation (3) is related to the Laplace transform of Φ_τ as follows:

$$\widetilde{L}_s = \frac{s\widetilde{\Phi}_s}{\mathbb{1} + \widetilde{\Phi}_s}. \tag{6}$$

Now, in order to explicitly write down L_τ one has to invert the Laplace transform $\widetilde{L}_s$. Note, however, that this might be very hard, due to the fact that $\widetilde{L}_s$ is a highly nontrivial function of s (possessing in general not only poles but also cuts in the complex s plane). It is therefore clear that even if one knows the solution $\rho_t = \Lambda_t \rho$, it is in general very difficult (if not practically impossible) to write down the corresponding non-Markovian Eq. (3). On the other hand, the knowledge of the (trace preserving and CP) solution Λ_t enables one to no longer care about the underlying equation. Let us look at an interesting example.

B. An example

The previous comments are best understood by looking at an example. Consider the pure decoherence model,

$$H = H_R + H_S + H_{SR}, \tag{7}$$

where H_R is the reservoir Hamiltonian,

$$H_S = \sum_n \epsilon_n P_n \quad (P_n = |n\rangle\langle n|) \tag{8}$$

the system Hamiltonian, and

$$H_{SR} = \sum_n P_n \otimes B_n \tag{9}$$

the interaction part, $B_n = B_n^\dagger$ being reservoirs operators. The initial product state $\rho \otimes \omega_R$ evolves according to the unitary evolution $e^{-iHt}(\rho \otimes \omega_R)e^{iHt}$, and by partial tracing with respect to the reservoir degrees of freedom, one finds for the evolved system density matrix

$$\rho_t = \mathrm{Tr}_R[e^{-iHt}(\rho \otimes \omega_R)e^{iHt}] = \sum_{n,m} c_{mn}(t) P_m \rho P_n, \tag{10}$$

where

$$c_{mn}(t) = \mathrm{Tr}(e^{-iZ_m t}\omega_R e^{iZ_n t}), \tag{11}$$

and the reservoir operators Z_n are defined by

$$Z_n = \epsilon_n \mathbb{I}_R + H_R + B_n. \tag{12}$$

Note that the matrix $c_{mn}(t)$ is semipositive definite and hence Eq. (10) defines the Kraus-Stinespring representation [20] of the CP map Λ_t:

$$\Lambda_t \rho = \sum_{n,m} c_{mn}(t) P_m \rho P_n. \tag{13}$$

The prescription (5) yields

$$\rho_t = \rho + \int_0^t \sigma_\tau \, d\tau, \tag{14}$$

$$\sigma_\tau = \Phi_\tau \rho = \dot{\rho}_\tau = \sum_{n,m} \dot{c}_{mn}(\tau) P_m \rho P_n, \tag{15}$$

and one very easily shows that $\mathrm{Tr}\, \sigma_\tau = 0$. The solution of the pure decoherence model can therefore be found without explicitly writing down the underlying master equation. What is (and needs to be) known is that ρ_t satisfies the non-Markovian master Eq. (3), but the construction of the corresponding memory kernel L_t is too formidable a task. Indeed, let us observe that due to the spectral property of Λ_t

$$\Lambda_t |m\rangle\langle n| = c_{mn}(t) |m\rangle\langle n|, \tag{16}$$

one obtains

$$L_t \rho = \sum_{n,m} \kappa_{mn}(t) P_m \rho P_n \tag{17}$$

for the corresponding generator, where the functions $\kappa_{mn}(t)$ are defined in terms of their Laplace transform as follows:

$$\widetilde{\kappa}_{mn}(s) = \frac{s\widetilde{c}_{mn}(s) - 1}{\widetilde{c}_{mn}(s)}. \tag{18}$$

Note that $c_{mm}(t) = 1$, and hence $\kappa_{mm}(t) = 0$. This condition guaranties that $L_t \mathbb{I} = 0$. However, the calculation of the off–diagonal elements $\kappa_{mn}(t)$ is in general not feasible.

Many similar examples are known in the physical literature, for example, in connection with the quantum Zeno effect. See [21] for a review on non-Markovian decay and [22] for its experimental observation. In what follows we shall therefore work directly with Λ_t and Eqs. (5) and (6), without detailing the features of the appropriate memory kernel L_t.

III. ASYMPTOTIC VS EQUILIBRIUM STATES

Let us now point out the crucial difference between Markovian and non-Markovian evolutions. Recall that a state ω is an equilibrium state for the (Markovian or non-Markovian) evolution Λ_t if

$$\lim_{t\to\infty} \Lambda_t \rho = \omega \quad \forall\, \rho. \tag{19}$$

One says that the evolution relaxes to ω and we shall assume for simplicity that ω is unique for the given Λ_t. On the other hand, a state ρ_0 is an invariant state for Λ_t if

$$\Lambda_t \rho_0 = \rho_0 \quad \forall\, t \geqslant 0. \tag{20}$$

Note that if Λ_t defines a semigroup (i.e., $\Lambda_t = e^{tL}$), then ρ_0 is invariant if $L\rho_0 = 0$. Clearly, for Markovian evolution the equilibrium state ω is always invariant. This is a straightforward consequence of the semigroup property $\Lambda_{s+t}(\omega) = \Lambda_t\big(\Lambda_s(\omega)\big)$ in the limit $s \to \infty$. However, this property is no longer true in the non-Markovian case, where the semigroup property cannot be used. Therefore, one may have non-Markovian evolutions relaxing to an asymptotic equilibrium state which is not invariant. In what follows, we shall analyze a few situations in order to explore the relaxing properties of non-Markovian evolutions.

A. A case study: convex combination of Markovian semigroups

Let $L_1, \ldots, L_n$ be a set of generators of Markovian equations of the type (1) and let $(p_1, \ldots, p_n)$ be a probability distribution ($\sum p_k = 1$). Then

$$\Lambda_t = \sum_{k=1}^{n} p_k\, e^{tL_k} \tag{21}$$

is by construction CP and satisfies (5) with

$$\Phi_t = \frac{d\Lambda_t}{dt} = \sum_{k=1}^{n} p_k\, L_k e^{tL_k}. \tag{22}$$

Actually, it is not difficult to conceive an evolution that is a convex combination of Markovian semigroups. Consider a system S living in $\mathcal{H}_S$ coupled to a reservoir R living in $\mathcal{H}_R$. (Actually, one may consider an arbitrary number N of reservoirs. In this case, $\mathcal{H}_R = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$.) Now, couple the composed S-R system to an n-level ancilla living in $\mathbb{C}^n$ and assume that the Hamiltonian has the form

$$H = \sum_{k=1}^{n} H_k \otimes P_k, \tag{23}$$

where $P_k = |k\rangle\langle k|$ ($|k\rangle$ is an orthonormal basis in the ancilla Hilbert space $\mathbb{C}^n$) and $H_k = H_k^\dagger$ are S-R operators . The unitary evolution generated by (23) reads

$$e^{-itH} = \sum_{k=1}^{n} e^{-itH_k} \otimes P_k; \tag{24}$$

hence, if the initial product state is $\rho \otimes \omega_R \otimes \sigma$, σ being a state of the ancilla, the reduced dynamics yields the following evolution for the system density operator:

$$\rho_t = \sum_{k=1}^{n} p_k \mathrm{Tr}_R[e^{-iH_k t}(\rho \otimes \omega_R) e^{iH_k t}], \tag{25}$$

where $p_k = \langle k|\sigma|k\rangle$. Standard weak coupling arguments lead to (21).

A convex combination (21) of Markovian semigroups is no longer a semigroup and satisfies the non-Markovian master Eq. (3). However, it can be very complicated to find the corresponding memory kernel. Observe that if for each k the corresponding Markovian evolution $\Lambda_t^{(k)} = e^{tL_k}$ possesses a unique equilibrium (and hence invariant) state ω_k, then Λ_t defined by (21) relaxes to the equilibrium state $\omega = \sum_{k=1}^{n} p_k \omega_k$. Note that ω_k need not be invariant for $\Lambda_t^{(l)}$ with $l \neq k$ (it is invariant if L_l and L_k commute). We stress that if each subgroup of ensemble members has its own Markovian decay process, toward its own equilibrium, then the global (non-Markovian) dynamics has a well-defined equilibrium (convex combination of Markovian equilibria) and hence the final state does not depend on the initial state (by definition of equilibrium). However, the equilibrium state ω needs not be invariant for the non-Markovian evolution governed by (21). That is, in general, $\Lambda_t \omega \neq \omega$, but of course, asymptotically, $\lim_{t\to\infty} \Lambda_t \omega = \omega$.

The simplest example of (21) corresponds to $L_1 = L$ and $L_2 = 0$, yielding the non-Markovian evolution

$$\Lambda_t = (1-p)e^{tL} + p\mathbb{1}, \tag{26}$$

that is, a mixture of a semigroup dynamics e^{tL} and the trivial one $\mathbb{1}\rho = \rho$. Equations (5) and (6) yield

$$\widetilde{L}_s = (1-p)L + \frac{p(1-p)\,L^2}{s-(1-p)L}, \tag{27}$$

which can be easily inverted:

$$L_t = 2(1-p)\delta(t)L + p(1-p)L^2 e^{t(1-p)L}. \tag{28}$$

Note the similarity with the Shabani-Lidar [14] memory kernel $L_t = Le^{tL}$ of the post-Markovian quantum master equation. In general, Le^{tL} does not lead to a CP dynamics. On the other hand, the kernel (28) generates a CP dynamics for arbitrary L. Formula (26) is an exceptional case: in general one cannot obtain a closed expression for the generator L_t. We stress that the non-Markovian dynamics (26) displays very peculiar features. Suppose that e^{tL} possesses an equilibrium (and hence invariant) state ω. It is clear that ω is still invariant for (26) but it is no longer an equilibrium state. Note that $L_t\omega = 0$ due to the fact that $L\omega = 0$. In conclusion, one has

$$\lim_{t\to\infty} \Lambda_t \rho = (1-p)\omega + p\rho, \tag{29}$$

which shows that ω cannot be reached asymptotically (unless we start with ω itself). Since, in general, a non-Markovian evolution is not relaxing, the asymptotic state strongly depends on the initial condition. This is the very essence of memory effects—the system remembers its initial state. We stress that this result is model independent. The only assumption is that L generates a relaxing Markovian semigroup. For example, one may take instead of the trivial generator $L_2 = 0$ the following one:

$$L_2' = -\gamma(\mathbb{1} - \mathcal{P}), \quad \gamma \geqslant 0, \tag{30}$$

where

$$\mathcal{P}\rho = \sum_n P_n \rho P_n \tag{31}$$

is a projector, with $P_n = |n\rangle\langle n|$, $|n\rangle$ being eigenvectors of ω. One has, therefore, $\mathcal{P}\omega = \omega$. Hence, the convex combination (21) yields the following formula:

$$\Lambda_t' = (1-p)e^{tL} + p[\mathcal{P} + e^{-\gamma t}(\mathbb{1} - \mathcal{P})]. \tag{32}$$

For $\gamma = 0$, $L_2' = L_2$ and one recovers (29). For $\gamma > 0$, the asymptotic formula (29) is replaced with

$$\lim_{t\to\infty} \Lambda_t' \rho = (1-p)\omega + p\,\mathcal{P}\rho. \tag{33}$$

Again, ω defines an invariant state for Λ_t'. However, Λ_t' is not relaxing and ω is not reachable (unless we start from it). Observe that the mixing parameter $p \in [0,1]$ in (26) and (32) measures in a sense the "non-Markovianity" of the evolution.

B. Quantum channel

We now look at a different example. Let

$$L_t = \kappa(t)(B - \mathbb{1}), \tag{34}$$

where B is a quantum channel (i.e., a trace preserving CP map) [12,18,19]. L_t generates a CP trace-preserving dynamics

Λ_t if the Laplace transform $\widetilde{\kappa}(s)$ satisfies

$$\widetilde{\kappa}(s) = \frac{s\widetilde{f}(s)}{1-\widetilde{f}(s)}, \tag{35}$$

where $f(t) \geqslant 0$ and $\int_0^\infty f(\tau)d\tau \leqslant 1$. Note that the corresponding Laplace transform of Λ_t reads

$$\widetilde{\Lambda}_s = \frac{1}{s}\frac{1-\widetilde{f}(s)}{\mathbb{I}-\widetilde{f}(s)B} \tag{36}$$

and in general cannot be inverted. However, even if we are not able to find Λ_t, we can easily study its asymptotic behavior. Indeed, using the well-known property of the Laplace transform

$$\lim_{t\to\infty} \Lambda_t = \lim_{s\to 0} s\widetilde{\Lambda}_s, \tag{37}$$

if all poles of $s\widetilde{\Lambda}_s$ are in the left-hand plane, one obtains from (36) the general asymptotic formula

$$\Lambda_\infty = \frac{1-\widetilde{f}(0)}{\mathbb{I}-\widetilde{f}(0)B}. \tag{38}$$

To study Λ_∞ in more detail, consider the spectral decomposition of B,

$$B\rho = \sum_{\alpha=0}^{d^2-1} b_\alpha F_\alpha \mathrm{Tr}(G_\alpha^\dagger \rho), \tag{39}$$

where d stands for the dimension of the system Hilbert space, and F_α and G_α define the biorthogonal damping basis of B. Suppose now that B possesses the unique invariant state ρ_0. This implies $F_0 = \rho_0$, $G_0 = \mathbb{I}$, and the corresponding eigenvalue $b_0 = 1$. One has, therefore,

$$\Lambda_\infty\rho = \rho_0 + \sum_{\alpha=1}^{d^2-1} \frac{1-\widetilde{f}(0)}{1-\widetilde{f}(0)b_\alpha} F_\alpha \mathrm{Tr}(G_\alpha^\dagger \rho). \tag{40}$$

Let us observe that if

$$\widetilde{f}(0) = \int_0^\infty f(\tau)d\tau = 1, \tag{41}$$

then $\Lambda_\infty\rho = \rho_0$; that is, the non-Markovian dynamics Λ_t is relaxing to the asymptotic equilibrium state ρ_0. However, if $\widetilde{f}(0) < 1$, then the dynamics is no longer relaxing and the asymptotic state $\Lambda_\infty\rho$ remembers the initial state ρ.

Consider for example $f(\tau) = \varepsilon\gamma e^{-\gamma\tau}$, with $\gamma > 0$ and $\varepsilon \in (0, 1]$. One has in this case

$$\widetilde{f}(0) = \varepsilon \leqslant 1, \tag{42}$$

and hence the parameter ε controls the asymptotic state $\Lambda_\infty\rho$. Let us observe that one can easily invert the Laplace transform (35) to obtain the following expression for the function $\kappa(t)$:

$$\kappa(t) = \varepsilon\gamma[2\delta(\tau) - \gamma(1-\varepsilon)e^{-\gamma(1-\varepsilon)\tau}]. \tag{43}$$

Observe that for $\varepsilon = 1$, one gets $\kappa(t) = 2\gamma\delta(t)$, which corresponds to the Markovian dynamics. Hence, the parameter $1-\varepsilon$ measures the deviation from the Markovianity.

This shows that non-Markovian evolutions are much more flexible. One can control the asymptotic behavior by controlling a single function of time $f(t)$ (for example, by controlling a single parameter ε). Note that in the Markovian case the evolution generated by (34) is given by

$$\Lambda_t^{\mathrm{M}}\rho = \sum_\alpha e^{\gamma b_\alpha t} F_\alpha \mathrm{Tr}(G_\alpha^\dagger \rho), \tag{44}$$

and hence it displays the characteristic exponential behavior $\exp(\gamma b_\alpha t)$. We stress that the Markovian evolution is relaxing to the unique invariant state ρ_0; that is, ρ_0 plays the role of equilibrium state for Λ_t^{M}. In the non-Markovian case, the evolution is relaxing only if $\widetilde{f}(0) = 1$. Note, however, that even if the evolution is relaxing, relaxation needs not be exponential.

C. Entanglement

It is clear that if e^{tL} describes the relaxing evolution of a composed system and its equilibrium state ω is separable, then all initially entangled states asymptotically become disentangled. This is no longer true for non-Markovian evolutions, such as (26) and (32). Whether the asymptotic state is separable may depend on the initial state as well. If one starts at time $t = 0$ with an entangled state ρ, the asymptotic state (29) or (33) might be entangled even if ω is separable. Moreover the system may consists of an arbitrary number of parties. For example, in the simplest case of a two-qubit system possessing an invariant (but not equilibrium) state ω which is maximally mixed (i.e., $\omega = \mathbb{I}/4$), Eq. (29) defines a mixed asymptotic state $(1-p)\mathbb{I}/4 + p\rho$. Hence, starting with a maximally entangled state $|\psi\rangle$ the dynamics (26) asymptotically approaches a Werner-like state,

$$\frac{1-p}{4}\mathbb{I} + p|\psi\rangle\langle\psi|, \tag{45}$$

which is entangled if $p > 1/3$ [23]. Hence, the "non-Markovianity parameter" p controls the entanglement of the asymptotic state.

Similarly, using the spectral resolution $\mathbb{I} = \sum_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$, with $|\psi_\alpha\rangle$ being the four Bell states, one finds that starting with an initial state ρ the non-Markovian dynamics (32) with an invariant state $\omega = \mathbb{I}/4$ asymptotically approaches the Bell-diagonal state

$$\sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|, \tag{46}$$

with $p_\alpha = (1-p)/4 + p\,\langle\psi_\alpha|\rho|\psi_\alpha\rangle$ depending upon the initial state ρ. It is well known that (46) is entangled if exactly one $p_\alpha > 1/2$. Again, p controls the separability properties of the asymptotic state (46).

Finally, consider the non-Markovian dynamics generated by the generator (34), where B is a quantum channel $B : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The simplest example of B is a projection defined by

$$B\rho = \sum_{m,n} P_{mn}\rho P_{mn}, \tag{47}$$

where $P_{mn} = |m \otimes n\rangle\langle m \otimes n| = P_m \otimes P_n$ are projectors onto the product vectors of the orthonormal basis in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, if ρ is a density operator of the bipartite system living in $\mathcal{H}_1 \otimes \mathcal{H}_2$, then representing ρ in the block form,

$$\rho = \sum_{m,n} |m\rangle\langle n| \otimes \widehat{\rho}_{mn}, \tag{48}$$

where $\widehat{\rho}_{mn}$ are operators in $\mathcal{B}(\mathcal{H}_2)$, one finds for the action of the projection B

$$B\rho = \sum_{m,n} (\widehat{\rho}_{nn})_{mm} P_m \otimes P_n. \tag{49}$$

It is easy to find the solution of the non-Markovian master equation

$$\Lambda_t = \left[1 - \int_0^t f(\tau)d\tau\right] \mathbb{1} + \int_0^t f(\tau)d\tau B, \tag{50}$$

where $f(\tau)$ is defined via formula (35). The density matrix has the following behavior: the diagonal blocks read

$$\widehat{\rho}_{mm}(t) = \left[1 - \int_0^t f(\tau)d\tau\right] \widehat{\rho}_{mm} + \int_0^t f(\tau)d\tau \sum_k (\widehat{\rho}_{kk})_{mn} P_k, \tag{51}$$

and the off-diagonal blocks

$$\widehat{\rho}_{mn}(t) = \left[1 - \int_0^t f(\tau)d\tau\right] \widehat{\rho}_{mn}, \tag{52}$$

for $m \neq n$. This shows that during the evolution the off-diagonal blocks are scaled by the factor $1 - \int_0^t f(\tau)d\tau$ and eventually disappear if $\int_0^\infty f(\tau)d\tau = 1$. The asymptotic state of the bipartite system reads

$$\Lambda_\infty \rho = [1 - \widetilde{f}(0)]\rho + \widetilde{f}(0)B\rho. \tag{53}$$

The asymptotic entanglement is controlled by $\widetilde{f}(0)$. It is therefore clear that if $\widetilde{f}(0) = 1$, then

$$\Lambda_\infty \rho = B\rho, \tag{54}$$

which is separable, being block-diagonal (the off-diagonal blocks disappear). Actually, due to formula (49), the asymptotic state $B\rho$ is not only block diagonal but even diagonal in the $|m \otimes n\rangle$ basis. It is, therefore, clear that in this case the state becomes separable in finite time and hence one encounters the sudden death of entanglement [4]. This happens in particular in the Markovian case [for a Markovian evolution, one has $1 - \int_0^t f(\tau)d\tau = e^{-\gamma t}$]. However, taking $f(\tau) = \varepsilon\gamma e^{-\gamma\tau}$, one has $\widetilde{f}(0) = \varepsilon$, and hence,

$$\Lambda_\infty \rho = (1 - \varepsilon)\rho + \varepsilon B\rho, \tag{55}$$

which shows that ε can control the asymptotic entanglement of ρ_∞. Starting from an entangled ρ, one may preserve entanglement forever by taking a large-enough deviation $1 - \varepsilon$ from Markovianity.

IV. CONCLUSIONS

In conclusion, we have shown that non-Markovian dynamics represent a completely new kind of quantum evolution. They are much more flexible than the Markovian ones and can lead to a completely novel behavior of the quantum system. In general, they provoke a modification of the characteristic exponential relaxation law known from Markovian evolutions. As a consequence, non-Markovianity entails new features of decoherence and relaxation to equilibrium. Interestingly, even if the evolution relaxes to an equilibrium state, this state need not be invariant. This can never happen in the Markovian case. Therefore, the noninvariance of equilibrium becomes a clear sign of non-Markovianity.

We have shown the asymptotic state of the system depends on the initial conditions, even if the non-Markovian dynamics possesses an invariant state. For composed systems, this implies that the asymptotic states can remember (and partially preserve) its initial entanglement. Hence, some residual entanglement can remain even in the remote future. Therefore, non-Markovian evolutions may avoid the phenomenon of sudden death of entanglement and can preserve entanglement forever. Our examples show that the asymptotic entanglement can be controlled by some characteristic parameters of the system in question (we called them non-Markovianity parameters). These model-independent conclusions have been illustrated by several examples and seem to pave the way toward a more general comprehension and practical exploitation of non-Markovian evolutions.

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