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Berry phase from a quantum Zeno effect

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Abstract

We exhibit a specific implementation of the creation of geometrical phase through the state-space evolution generated by the dynamic quantum Zeno effect. That is, a system is guided through a closed loop in Hilbert space by means a sequence of closely spaced projections leading to a phase difference with respect to the original state. Our goal is the proposal of a specific experimental setup in which this phase could be created and observed. To this end we study the case of neutron spin, examine the practical aspects of realizing the 'projections', and estimate the difference between the idealized projections and the experimental implementation. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The effect of the observer in quantum mechanics is perhaps nowhere more dramatic than in the collection of phenomena loosely (and casually) known as the 'quantum Zeno effect'. This was first formulated by von Neumann [1,2], and is deeply rooted in fundamental features of the temporal behavior of quantum systems [3]. During the last decade there has been much interest in this issue, mainly because of an idea due to Cook [4], who proposed using two-level systems to check this effect, and the subsequent experiment performed by Itano et al. [5]. New experiments were proposed, based on the physics of the simplest of two-level systems: Neutron spin and photon polarization [6,7].

Most of the referenced papers deal with what might be called the 'static' version of the quantum Zeno effect. However, the most striking action of the observer is not only to stop time evolution (e.g., by repeatedly checking if a system has decayed), but to *guide* it. In this article we will be concerned with a 'dynamical' version of the phenomenon: we will show how guiding a system through a closed loop in its state space (projective Hilbert space) leads to a geometrical phase [8–12]. This was predicted on general grounds [13], but here we use a specific implementation on a spin system [14] and propose a particular experimental context in which to see this effect. It is remarkable that the Berry phase that is discussed below is due to measurements only: no Hamiltonian is needed.

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2. Forcing the pot to boil

We summarize the main features of the quantum Zeno effect (QZE). Prepare a quantum system in some initial state $\psi(0)$. In time dt, by the Schrödinger equation, its phase changes by $O(dt)$ while the absolute value of its scalar product with the initial state changes by $O(dt^2)$.

The *dynamical* quantum Zeno effect exploits the above features and forces the evolution in an arbitrary direction by a series of repeated measurements: Let ψ evolve with the Hamiltonian H, so that in the absence of observations its evolution would be $\psi(T) = \exp(-iHT)\psi(0)$ (we take $\hbar = 1$ throughout). Let there be a family of states ϕ_k, $k = 0,1,\dots,N$, such that $\phi_0 = \psi(0)$, and such that successive states differ little from one another (i.e., $|\langle\phi_{k+1}|\phi_k\rangle|$ is nearly 1). Now let $\delta T = T/N$ and at $T_k = k\delta T$ project the evolving wave function on ϕ_k. Then for sufficiently large N, $\psi(T) \approx \phi_N$. [The usual QZE is the special case $\phi_k = \phi_0 (= \psi(0))\,\forall\, k$.]

In the following we consider an experiment involving a neutron spin. It should be clear, however, that our proposal is valid for any system with the same two-level structure.

2.1. Evolution with no Hamiltonian

Assume first that there is *no* Hamiltonian acting on the system: one can think, for instance, of a neutron crossing a region where no magnetic field is present. The time-evolution is due to measurement only.

The system starts with spin up along the z-axis and is projected on the family of states

$$\phi_k \equiv \exp(-i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n})\begin{pmatrix}1\\0\end{pmatrix}$$

$$\text{with } \theta_k \equiv \frac{ak}{N}, \quad k = 0,\dots,N, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the vector of the Pauli matrices and $\boldsymbol{n} = (n_x, n_y, n_z)$ a unit vector (independent of k).

We assume that the system evolves for a time T with projections at times $T_k = k\delta T$ ($k = 1,\dots,N$ and $\delta T = T/N$). The final state is $\left[\phi_0 = \begin{pmatrix}1\\0\end{pmatrix}\right]$

$$\begin{aligned}
|\psi(T)\rangle &= |\phi_N\rangle\langle\phi_N|\phi_{N-1}\rangle\cdots\langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_0\rangle \\
&= |\phi_N\rangle\left(\cos\frac{a}{N} + in_z\sin\frac{a}{N}\right)^N \\
&= \cos^N\!\left(\frac{a}{N}\right)\left(1 + in_z\tan\frac{a}{N}\right)^N|\phi_N\rangle \\
&\overset{N\to\infty}{\to} \exp(ian_z)|\phi_N\rangle \\
&= \exp(ian_z)\exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n})|\phi_0\rangle. \qquad (2.2)
\end{aligned}$$

Therefore, as $N \to \infty$, $\psi(T)$ is an eigenfunction of the final projection operator P_N, with unit norm. If $\cos\Theta \equiv n_z$ and $a = \pi$,

$$\begin{aligned}
\psi(T) &= \exp(i\pi\cos\Theta)(-1)\phi_0 \\
&= \exp[-i\pi(1-\cos\Theta)]\phi_0 \\
&= \exp(-i\Omega/2)\phi_0, \qquad (2.3)
\end{aligned}$$

where Ω is the solid angle subtended by the curve traced by the spin during its evolution. The factor $\exp(-i\Omega/2)$ is a Berry phase and it is due only to measurements (the Hamiltonian is zero). Notice that no Berry phase appears in the usual quantum Zeno context, namely when $\phi_k \propto \phi_0\,\forall\, k$, because in that case $a = 0$ in (2.2).

To provide experimental implementation of the mathematical process just described, one could (in principle) let a neutron spin evolve in a field-free region of space. With no further tinkering, the spin state would not change. However, suppose we place spin filters sequentially projecting the neutron spin onto the states of Eq. (2.1), for $k = 0,\dots,N$. Thus the neutron spin is forced to follow another trajectory in spin space. The essence of the mathematical demonstration just provided is that while N measurements are performed, the norm of wave function that is absorbed by the filters is $N\cdot O(1/N^2) = O(1/N)$. For $N \to \infty$, this loss is negligible. Meanwhile, as a result of these projections, the trajectory of the spin (in its space) is a cone whose symmetry axis is $\boldsymbol{n}$. By suitably matching the parameters, the spin state

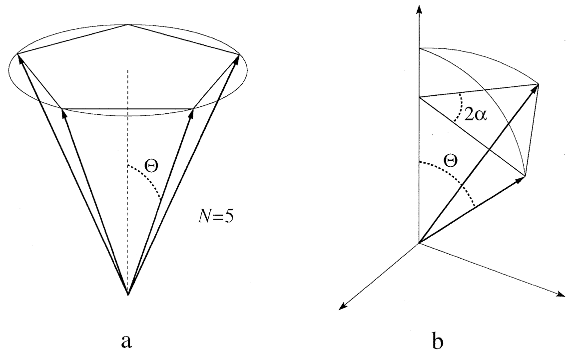

Fig. 1. a) Spin evolution due to $N=5$ measurements. b) Solid angles.

can be forced back to its initial state after time T [14].

It is interesting to look at the process (2.2) for N finite. The spin goes back to its initial state after describing a regular polygon on the Poincaré sphere, as in Fig. 1a. After $N(<\infty)$ projections the final state is

$$|\psi(T)\rangle=\left(\cos\frac{a}{N}+in_z\sin\frac{a}{N}\right)^N \times\exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n})|\phi_0\rangle. \qquad (2.4)$$

For $a=\pi$ the spin describes a closed path and

$$\begin{aligned}|\psi(T)\rangle&=\left(\cos\frac{\pi}{N}+in_z\sin\frac{\pi}{N}\right)^N\exp(-i\pi)|\phi_0\rangle\\&=\left(\cos^2\frac{\pi}{N}+n_z^2\sin^2\frac{\pi}{N}\right)^{N/2}\\&\quad\times\exp\left(iN\arctan\left(n_z\tan\frac{\pi}{N}\right)\right)\\&\quad\times\exp(-i\pi)|\phi_0\rangle.\end{aligned} \qquad (2.5)$$

The first factor in the far r.h.s. accounts for the probability loss (N is finite and there is no QZE). We can rewrite (2.5) in the following form:

$$|\psi(T)\rangle=\rho_N\exp(-i\beta_N)|\phi_0\rangle, \qquad (2.6)$$

where

$$\rho_N=\left(\cos^2\frac{\pi}{N}+n_z^2\sin^2\frac{\pi}{N}\right)^{N/2}, \qquad (2.7)$$

$$\beta_N=\pi-N\arctan\left(\cos\Theta\tan\frac{\pi}{N}\right). \qquad (2.8)$$

In the 'continuous measurement' limit (QZE), we have

$$\rho=\lim_{N\to\infty}\rho_N=1,$$
$$\beta=\lim_{N\to\infty}\beta_N=\pi(1-\cos\Theta)=\frac{\Omega}{2}, \qquad (2.9)$$

where Ω is the solid angle subtended by the circular path, viewed at an angle Θ (see Fig. 1a). We recover therefore the result (2.3).

The relation between the solid angle and the geometrical phase is valid also with a finite number of polarizers N. Indeed, it is straightforward to show that the solid angle subtended by an isosceles triangle with vertex angle equal to 2α (Fig. 1b) has the value

$$\Omega_{2\alpha}=2\alpha-2\arctan(\cos\Theta\tan\alpha). \qquad (2.10)$$

Hence if the polarizers are equally rotated of an angle $2\pi/N$, the spin describes a regular N-sided polygon, whose solid angle is

$$\begin{aligned}\Omega_{(N)}&=N\Omega_{2\pi/N}=2\pi-2N\arctan\left(\cos\Theta\tan\frac{\pi}{N}\right)\\&=2\beta_N,\end{aligned} \qquad (2.11)$$

where we used the definition (2.8). This result is of course in agreement with other analyses [15] based on the Pancharatnam connection [8].

The above conclusion can be further generalized to the general case of an arbitrary (not necessarily regular) polygon. Indeed, if the polarizers are rotated at (relative) angles α_n with $n=0,\dots,N$, so that

$$\sum_{n=1}^{N}2\alpha_n=2\pi, \qquad (2.12)$$

the solid angle is

$$\Omega'_{(N)}=\sum_{n=1}^{N}\Omega_{2\alpha_n}=2\pi-2\sum_{n=1}^{N}\arctan(\cos\Theta\tan\alpha_n). \qquad (2.13)$$

This is also twice the Berry phase. Notice that if all $\alpha_n\to0$ as $N\to\infty$ one again obtains the limit (2.3):

$$\Omega'=\lim_{N\to\infty}\Omega'_N=2\pi-2\lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n\cos\Theta=\Omega. \qquad (2.14)$$

We emphasize that these predictions for the $N<\infty$ case are not trivial from the physical point of view. The above phases are computed by assuming that, during a 'projection' à la von Neumann, the spin follows a geodesics on the Poincaré sphere. The mathematics of the projection has no such assumptions. The 'postulate's' only job is to relate all this projection formalism to measurements.

2.2. Evolution with a non-zero Hamiltonian

Let us now consider the effect of a non-zero Hamiltonian

$$H=\mu\boldsymbol{\sigma}\cdot\boldsymbol{b}, \tag{2.15}$$

where $\boldsymbol{b}=(b_x,b_y,b_z)$ is a unit vector, in general different from $\boldsymbol{n}$. One can think of a neutron spin in a magnetic field. See Fig. 2.

If the system starts with spin up it would have the following – undisturbed – evolution:

$$\psi(t)=\exp(-i\mu t\boldsymbol{\sigma}\cdot\boldsymbol{b})\begin{pmatrix}1\\0\end{pmatrix}. \tag{2.16}$$

Now let the system evolve for a time T with projections at times $T_k=k\delta T$ $(k=1,\dots,N$ and $\delta T=T/N)$ and Hamiltonian evolution in between. Defining $P_0\equiv|\phi_0\rangle\langle\phi_0|=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, the 2×2 projection operator at stage-k is

$$P_k=|\phi_k\rangle\langle\phi_k|=\exp(-i\theta_k\boldsymbol{\sigma}\cdot\boldsymbol{n})P_0\exp(i\theta_k\boldsymbol{\sigma}\cdot\boldsymbol{n}) \tag{2.17}$$

and the state evolves to

$$\psi(T)=\left[\prod_{k=1}^{N}\left[P_k\exp(-i\mu\delta T\boldsymbol{\sigma}\cdot\boldsymbol{b})\right]\right]\begin{pmatrix}1\\0\end{pmatrix}, \tag{2.18}$$

where here and in subsequent expressions a time-ordered product is understood [with earlier times (lower k) to the right]. Using $P_0^2=P_0$, Eq. (2.18) can be rewritten

$$\psi(T)=\exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n})\left[\prod_{k=1}^{N}B_k\right]\begin{pmatrix}1\\0\end{pmatrix}, \tag{2.19}$$

with

$$B_k\equiv P_0\exp(i\theta_k\boldsymbol{\sigma}\cdot\boldsymbol{n})\exp(-i\mu\delta T\boldsymbol{\sigma}\cdot\boldsymbol{b})\times\exp(-i\theta_{k-1}\boldsymbol{\sigma}\cdot\boldsymbol{n})P_0 \tag{2.20}$$

$(\theta_0\equiv0)$. The computation of B_k requires a bit of SU(2) manipulation. By using

$$[\boldsymbol{\sigma}\cdot\boldsymbol{A},\boldsymbol{\sigma}\cdot\boldsymbol{B}]=2i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{B} \tag{2.21}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B})(\boldsymbol{\sigma}\cdot\boldsymbol{A})=2(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{\sigma}\cdot\boldsymbol{A}-(\boldsymbol{A}\cdot\boldsymbol{A})\boldsymbol{\sigma}\cdot\boldsymbol{B}, \tag{2.22}$$

valid for c-number $\boldsymbol{A}$ and $\boldsymbol{B}$, one gets

$$\exp(i\theta\boldsymbol{\sigma}\cdot\boldsymbol{n})\boldsymbol{\sigma}\cdot\boldsymbol{b}\exp(-i\theta\boldsymbol{\sigma}\cdot\boldsymbol{n})=\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{b}}, \tag{2.23}$$

with

$$\tilde{\boldsymbol{b}}(\theta)\equiv\boldsymbol{b}\cos2\theta+\boldsymbol{n}(\boldsymbol{b}\cdot\boldsymbol{n})(1-\cos2\theta)+\boldsymbol{b}\times\boldsymbol{n}\sin2\theta, \tag{2.24}$$

which is the vector $\boldsymbol{b}$ rotated by 2θ about the $\boldsymbol{n}$-axis. The calculation of B_k is now straightforward:

$$\begin{aligned}B_k&=P_0\exp(i\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n})\exp\big(-i\mu\delta T\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{b}}(\theta_{k-1})\big)\\ P_0&=P_0\big(1+i\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}-i\mu\delta T\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{b}}(\theta_k)\big)P_0\\&\quad+O(1/N^2),\end{aligned} \tag{2.25}$$

where $\delta\theta=\theta_{k+1}-\theta_k$ is k-independent. Second order terms in $1/N$ drop out when the product (2.19) is computed for $N\to\infty$, so that

$$\begin{aligned}\prod_{k=1}^{N}B_k&=\prod_{k=1}^{N}P_0\big(1+i\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}-i\mu\delta T\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{b}}(\theta_k)\big)P_0\\&=\prod_{k=1}^{N}\Big\{P_0+iP_0\big(\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}\\&\qquad-\mu\delta T\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{b}}(\theta_k)\big)P_0\Big\}\\&=\prod_{k=1}^{N}P_0\Big\{1+i\big[\delta\theta n_z-\mu\delta T\tilde{b}_z(\theta_k)\big]\Big\}\\&=P_0\exp\left\{i\sum_{k=1}^{N}\big(\delta\theta n_z-\mu\delta T\tilde{b}_z(\theta_k)\big)\right\},\end{aligned} \tag{2.26}$$

where we have used $P_0\sigma_xP_0=P_0\sigma_yP_0=0$ and $P_0\sigma_zP_0=P_0$. The continuum limit can be computed

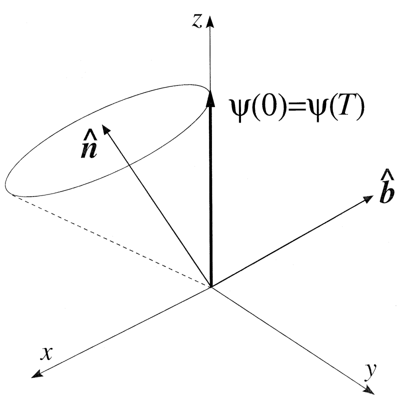

Fig. 2. Spin evolution with measurements and non-zero Hamiltonian.

by letting the summations in (2.26) become integrals in dT and $d\theta$. Moreover, $dT/d\theta = T/a$, which enables one to change integration variable and get for the '(1,1)' component of $\prod_{k=1}^{N} B_k$ (all other components being zero)

$$\begin{aligned}
&\exp\left(in_z \int_0^a d\theta - i\mu \frac{T}{a} \int_0^a \left[b_z \cos 2\theta \right.\right. \\
&\quad \left.\left. + (\boldsymbol{b}\cdot\boldsymbol{n}) n_z (1-\cos 2\theta) + (\boldsymbol{b}\times\boldsymbol{n})_z \sin 2\theta \right] d\theta \right) \\
&\quad = \exp\left(in_z a - i\mu \frac{T}{a} \left[b_z \frac{\sin 2a}{2} \right.\right. \\
&\qquad + (\boldsymbol{b}\cdot\boldsymbol{n}) n_z \left(a - \frac{\sin 2a}{2} \right) \\
&\qquad \left.\left. + (\boldsymbol{b}\times\boldsymbol{n})_z \frac{1-\cos 2a}{2} \right] \right),
\end{aligned} \tag{2.27}$$

The final state is an eigenstate of P_N with unit norm, *independent* of the Hamiltonian H:

$$\begin{aligned}
\psi(T) = &\exp\left(-i\mu \frac{T}{a} \left[b_z \frac{\sin 2a}{2} \right.\right. \\
&\quad + (\boldsymbol{b}\cdot\boldsymbol{n}) n_z \left(a - \frac{\sin 2a}{2} \right) \\
&\quad \left.\left. + (\boldsymbol{b}\times\boldsymbol{n})_z \frac{1-\cos 2a}{2} \right] \right) \\
&\quad \times \exp(ian_z - ia\boldsymbol{\sigma}\cdot\boldsymbol{n}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.28}$$

The first factor in (2.28) is obviously the 'dynamical phase'. Note that up to a phase, $\psi(t)$ is just ϕ_k, with $k = tN/T$. Therefore

$$\begin{aligned}
&\int_0^T \langle \psi(t) | H | \psi(t) \rangle dt \\
&\quad = \frac{T}{a} \int_0^a \langle \phi_0 | \exp(i\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}) \mu \boldsymbol{\sigma}\cdot\boldsymbol{b} \\
&\qquad \times \exp(-i\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}) | \phi_0 \rangle d\theta \\
&\quad = \mu T \left[b_z \frac{\sin 2a}{2a} + (\boldsymbol{b}\cdot\boldsymbol{n}) n_z \left(1 - \frac{\sin 2a}{2a} \right) \right. \\
&\qquad \left. + (\boldsymbol{b}\times\boldsymbol{n})_z \frac{1-\cos 2a}{2a} \right],
\end{aligned} \tag{2.29}$$

because the phases drop out in the above sandwich. It follows that the remaining phase in (2.28), when the spin goes back to its initial state, is the geometrical phase. When $a = \pi$

$$\psi(T) = \exp(-i\Omega/2) \exp(-i\mu T (\boldsymbol{b}\cdot\boldsymbol{n}) n_z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.30}$$

where Ω is the solid angle subtended by the curve traced out by the spin, as in (2.3), and $\mu T(\boldsymbol{b}\cdot\boldsymbol{n}) n_z$ yields the dynamical phase, as can also be seen by direct computation of (2.29). We remark that if time ordered products are looked upon as path integrals [16], then our above demonstration is effectively a path integral derivation of the geometrical phase.

A practical implementation of the process just described would involve an experimental setup similar to the one described after Eq. (2.3), but with a magnetic field whose action on the spin is described by the Hamiltonian (2.15). If the neutron were to evolve *only* under the action of the Hamiltonian, its spin would precess around the magnetic field. However, the sequence of spin filters, which project the neutron spin onto the states (2.1), compel the spin to follow the same trajectory as in the previous case [Eq. (2.2)], i.e. a cone whose symmetry axis is $\boldsymbol{n}$. As above, the spin acquires a geometrical phase, but now there is a dynamical phase as well.

2.3. A particular case

It is instructive to look at a particular case of (2.28)–(2.30). We first note that if $\mu = 0$ in (2.28) we recover (2.2). Now let $\boldsymbol{b} = \boldsymbol{n}$. In this situation the projectors and the Hamiltonian yield the same trajectory in spin space (although, as will be seen, at different rates). If $\mu = 0$ (so that $H = 0$), the spin evolution is only due to the projectors and the final result was computed in (2.3)

$$\psi(T) = \exp(-i\Omega/2) \phi_0 . \tag{2.31}$$

If, on the other hand, there is a nonvanishing Hamiltonian (2.15), but *no* projectors are present, a cyclic evolution of the spin is obtained for $\mu T = \pi$. The calculation is elementary and yields

$$\psi(T) = \exp(-i\pi) \phi_0 . \tag{2.32}$$

Table 1
Phases for cyclic spin evolutions

	$H=0$ and projections	$H=\mu\boldsymbol{\sigma}\cdot\boldsymbol{b}$ no projections	$H=\mu\boldsymbol{\sigma}\cdot\boldsymbol{b}$ and projections
$\phi_{\rm geom}$	$\Omega/2$	$\Omega/2$	$\Omega/2$
$\phi_{\rm dyn}$	0	$\pi-\Omega/2$	$\mu T n_z$
$\phi_{\rm tot}=\phi_{\rm geom}+\phi_{\rm dyn}$	$\Omega/2$	$\pi(=\mu T)$	$\Omega/2+\mu T n_z$
	cyclic evolution due to projections	cyclic evolution due to field	cyclic evolution due to projections

Observe that the dynamical phase in this case is $[\mu T=\pi, \boldsymbol{b}=\boldsymbol{n}$ and $a=\pi$ in Eq. (2.29)]

$$\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt = \pi n_z = \pi[1-(1-n_z)] = \pi - \Omega/2. \tag{2.33}$$

Therefore, the 'π' phase in (2.32) can be viewed, à la Aharonov and Anandan [13], as the sum of a geometrical ($\Omega/2$) and a dynamical ($\pi-\Omega/2$) contribution.

Now let both the Hamiltonian and the projectors be present. From Eq. (2.30), one gets

$$\psi(T) = \exp(-i\Omega/2)\exp(-i\mu T n_z)\begin{pmatrix}1\\0\end{pmatrix}. \tag{2.34}$$

Notice that the value of μ is now arbitrary, so that μT is not necessarily equal to π (the cyclic evolution of the spin is due to the projectors, not to the Hamiltonian). When $\mu T<\pi$, the projections are too 'fast' and do not yield (2.32). On the other hand, when $\mu T>\pi$, the projections are too slow and supply less phase, in comparison with Eq. (2.32). Only in the case $\mu T=\pi$ do the projections yield the right phase in (2.32). Their presence is superfluous in this case: one would obtain exactly the same vector and the same phase without them. Our conclusions are summarized in Table 1. In some sense, one may say that the Hamiltonian dynamics provides a 'natural clock' for the phase of the wave function.

3. A gedanken experiment

An experimental implementation with neutrons would be difficult because it would involve putting a QZE set-up inside an interferometer in order to measure phase. We therefore restrict ourselves to a 'gedanken experiment' based on the use of ^{3}He as a neutron polarization filter [17]. It is well known [18] that Helium 3 is 'black' to neutrons but *polarized* ^{3}He only absorbs one spin state of a neutron beam – hence acts as a 50% absorber of a beam; the rest of it emerges fully polarized. In practice an external magnetic field is used to maintain the polarization axis of the ^{3}He. If this external bias field were to be given a slow twist along a longitudinal axis, the state of polarization of the ^{3}He should follow the direction of the twist. A neutron beam propagating through a cell of high-pressure polarized ^{3}He along an axis aligned with the direction of twist will become fully polarized and should develop a Berry phase according to the argument of the previous section.

From an experimental perspective a significant problem is that we so far lack a notion of slowness (as when we speak of 'slow twist' of the B field). In the previous calculation, it is implicitly assumed that θ changes more slowly than t (time): in other words, the relaxation processes in the ^{3}He are given enough time (are fast enough) to function as a polarizer. A full treatment of this problem should therefore describe the physics of the projection process. We now tackle this issue and see that the notion of slowness can be given quantitative meaning in terms of a condition for adiabaticity.

In practice, the absorption of the non-selected spin state occurs over a finite distance, of the order of one or two centimeters. This situation can be modeled via the following family of effective (non-hermitian) Hamiltonians:

$$H_k = -iV|\phi_k^\perp\rangle\langle\phi_k^\perp|, \tag{3.1}$$

where V is a real constant and

$$\phi_k^\perp \equiv \exp(-i\theta_k\boldsymbol{\sigma}\cdot\boldsymbol{n})\begin{pmatrix}0\\1\end{pmatrix}$$

$$\text{with } \theta_k \equiv \frac{ak}{N}, \quad k=0,\dots,N. \tag{3.2}$$

Note that $\langle\phi_k|\phi_k^\perp\rangle=0$ [see Eq. (2.1)]. We first assume, for simplicity, that no external (^{3}He aligning) magnetic field is present. We define

$$P_k^{\perp} \equiv |\phi_k^{\perp}\rangle\langle\phi_k^{\perp}| = \exp(-i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n}) P_0^{\perp} \exp(i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n}); \quad \left(P_0^{\perp} = |\phi_0^{\perp}\rangle\langle\phi_0^{\perp}|\right). \tag{3.3}$$

Obviously $P_k^{\perp} = 1 - P_k$, where P_k was defined in (2.17). The evolution engendered by the above Hamiltonian reads

$$e^{-iH_k\tau} = P_k + \epsilon P_k^{\perp} = \exp(-i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n}) \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \exp(i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n}) \equiv P'_k, \tag{3.4}$$

where (inserting $\hbar$)

$$\epsilon \equiv e^{-V\tau/\hbar} \tag{3.5}$$

is a parameter yielding an estimate of the efficiency of the polarizer. One can estimate a minimal value for V: for a thermal neutron (speed $v \simeq 2000\,\mathrm{m/s}$) and an absorption length ℓ on the order of 1 cm for the wrong-spin component, one gets $\tau = \ell/v \simeq 5\mu$s and one obtains a good polarizer for $V > \hbar/\tau \simeq 10^{-29}$ J $\simeq 10^{-7}$ meV.

The evolution can be computed by using the technique of Section 2 ($\sqrt{P'_0} = P_0 + \epsilon^{1/2} P_0^{\perp}$):

$$\psi'(T) = \exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n}) \sqrt{P'_0} \left[\prod_{k=1}^{N} B'_k\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.6}$$

with $T = N\tau$ and

$$\prod_{k=1}^{N} B'_k = \prod_{k=1}^{N} \sqrt{P'_0}(1 + i\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n})\sqrt{P'_0} = \prod_{k=1}^{N} P'_0 + i\sqrt{P'_0}(\delta\theta\boldsymbol{\sigma}\cdot\boldsymbol{n})\sqrt{P'_0} = \begin{pmatrix} 1 + i\delta\theta n_z & i\delta\theta\epsilon^{1/2} n_- \\ i\delta\theta\epsilon^{1/2} n_+ & \epsilon(1 - i\delta\theta n_z) \end{pmatrix}^N, \tag{3.7}$$

where $n_{\pm} \equiv n_x \pm i n_y$. The evaluation of the above matrix product when $N \to \infty$ is lengthy but straightforward. One gets

$$\psi'(T) = \exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n}) \mathscr{M} \phi_0, \tag{3.8}$$

where

$$\mathscr{M} = \frac{e^{-ab}}{\Delta} \times \begin{pmatrix} \Delta\,\mathrm{ch}(a\Delta) + (b + in_z)\,\mathrm{sh}(a\Delta) & in_-\,\mathrm{sh}(a\Delta) \\ in_+\,\mathrm{sh}(a\Delta) & \Delta\,\mathrm{ch}(a\Delta) - (b + in_z)\,\mathrm{sh}(a\Delta) \end{pmatrix}, \tag{3.9}$$

with

$$b = \frac{VT}{2a\hbar}, \quad \Delta = \sqrt{b^2 + 2ibn_z - 1}\,. \tag{3.10}$$

We are interested in the limit of large $b = VT/2a\hbar$. Indeed, larger values of b correspond to more ideal polarizers. In fact $\gamma = V/\hbar$ represents the absorption rate of the wrong component of the spin, while $\omega = 2a/T$ is the angular velocity of precession (the spin describes an angle of $2a$ in time T). The parameter $b = \gamma/\omega$ is the ratio of these two quantities. Large values of b imply

$$\gamma \gg \omega, \tag{3.11}$$

i.e., an absorption rate much larger than the velocity of precession. In other words, the spin rotation must be sufficiently slow to allow the absorption of the wrong component of the spin. By introducing the neutron speed v, one can define the absorption length $\ell = v/\gamma = v\hbar/V$ and the length covered by the neutron while rotating for 1 rad, $L = v/\omega = vT/2a$. Hence (3.11) reads

$$L \gg \ell. \tag{3.12}$$

These are all conditions of adiabaticity.

In the large b limit, using the definition (3.10), (3.9) becomes

$$\mathscr{M} = \frac{e^{a(\Delta - b)}}{2\Delta} \begin{pmatrix} \Delta + b + in_z & in_- \\ in_+ & \Delta - b - in_z \end{pmatrix} + \mathrm{O}(e^{-2ab}) = \exp(ian_z) \begin{pmatrix} 1 - a\dfrac{1 - n_z^2}{2b} & i\dfrac{n_-}{2b} \\ i\dfrac{n_+}{2b} & 0 \end{pmatrix} + \mathrm{O}\left(\frac{1}{b^2}\right). \tag{3.13}$$

Remembering the definition of b in (3.10), one gets

$$\mathscr{M} = \exp(ian_z) \begin{pmatrix} 1 + \dfrac{\hbar a^2(n_z^2 - 1)}{VT} & i\dfrac{\hbar a n_-}{VT} \\ i\dfrac{\hbar a n_+}{VT} & 0 \end{pmatrix} + \mathrm{O}\left(\left(\frac{2a\hbar}{VT}\right)^2\right) \to \exp(ian_z) P_0, \quad \text{when } \frac{VT}{2a\hbar} \to \infty. \tag{3.14}$$

The above formula yields the first corrections to an ideal, purely adiabatic evolution. Basically, the system is projected on slightly different directions, thereby rotating in spin space. But if the system 'on its own' (i.e., through its dynamics) manages to rotate significantly between projections, then more will be absorbed on the next projection and it will not follow the rotating field, at least not without loss of probability (or intensity).

It is interesting to note that the same result can be obtained by considering a continuous version of the effective Hamiltonian (3.1)

$$H(t) = -iVP^{\perp}(t) = -iVU^{\dagger}(t)\,P_0^{\perp}\,U(t), \tag{3.15}$$

where

$$U(t) = \exp\left(i\frac{a}{T}t\,\boldsymbol{\sigma}\cdot\boldsymbol{n}\right) \tag{3.16}$$

is a unitary operator (rotation). The state vector $\psi(t)$ satisfies the Schrödinger equation

$$i\partial_t\psi(t) = H(t)\psi(t). \tag{3.17}$$

Consider now the following rotated vector:

$$\tilde{\psi}(t) = U(t)\psi(t). \tag{3.18}$$

It is easy to prove that it satisfies the equation

$$i\partial_t\tilde{\psi}(t) = \tilde{H}\tilde{\psi}(t), \tag{3.19}$$

where

$$\begin{aligned}\tilde{H} &= i\dot{U}(t)U^{\dagger}(t) + U(t)H(t)U^{\dagger}(t)\\ &= -\frac{a}{T}\boldsymbol{\sigma}\cdot\boldsymbol{n} - iVP_0^{\perp}\end{aligned} \tag{3.20}$$

is independent of t. One then gets

$$\begin{aligned}\psi(t) &= U^{\dagger}(t)\tilde{\psi}(t)\\ &= \exp\left(-i\frac{a}{T}t\,\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\exp(-i\tilde{H}t)\,\psi(0),\end{aligned} \tag{3.21}$$

where

$$\tilde{H}T = -a\boldsymbol{\sigma}\cdot\boldsymbol{n} - iVTP_0^{\perp} = -aM,$$
$$M = \begin{pmatrix} n_z & n_- \\ n_+ & -n_z + i2b \end{pmatrix}, \tag{3.22}$$

b being defined in (3.10). Hence one obtains

$$\exp(-i\tilde{H}T) = \exp(iaM) = \mathscr{M} \tag{3.23}$$

and (3.21) yields (3.8). Observe that

$$\tilde{H} = -\omega\frac{\boldsymbol{\sigma}\cdot\boldsymbol{n}}{2} - i\gamma P_0^{\perp}, \tag{3.24}$$

from which it is apparent the previous interpretation of the coefficients ω and γ.

The above calculation was performed by assuming that no external field is present. However, we do need an external B field, in order to align ^{3}He. Its effect can be readily taken into account by noticing that, when the neutron crosses the region containing polarized ^{3}He, if the conditions for adiabaticity are satisfied, the neutron spin will always be (almost) parallel to the direction of ^{3}He and therefore to the direction of the magnetic field. The resulting dynamical phase is therefore trivial to compute and reads $\phi_{\rm dyn} \simeq \mu BT/\hbar$. In order to obtain the geometric phase in a realistic experiment, such a dynamical phase should be subtracted from the total phase acquired by the neutron during its interaction with ^{3}He. Incidentally, notice that this is experimentally feasible: one can take into account the contribution of a large dynamical phase due to the magnetic field and neatly extract a small Berry phase [19]. The novelty of our proposal consists in the introduction of polarizing ^{3}He to force the neutron spin to follow a given trajectory is spin space.

An alternative realization relies on a set of discrete ^{3}He polarization filters with progressively tilted polarization axes, as a finite-difference approximation to the system discussed above. Such a system would be a neutron analog of a set of polaroid filters with progressively tilted axes through which a photon beam propagates with little or no loss (in the limit of small angles) as proposed by Peres [2]. However, in the case discussed in this Letter, the axes of the neutron polarizers need not belong to a single plane and the neutron can acquire a Berry phase as well as change in polarization direction.

After completion of this paper we learned of interesting related work by Berry and Klein and by Pati and Lawande. See Refs. [20,21].

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