

All You Ever Wanted to Know About the Quantum Zeno Effect in 70 Minutes

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Abstract. This is a primer on the quantum Zeno effect, addressed to students and researchers with no previous knowledge on the subject. The prerequisites are the Schrödinger equation and the von Neumann notion of projective measurement.

1. Introduction and Motivation

The evolution of an unstable quantum system is characterized by three distinct regimes [1, 2]: a short-time region, where the decay is quadratic, an intermediate region, during which the exponential law sets in, and a long-time region, governed by a power law. A sketch (not in scale!) of such an evolution is given in Fig. 1.

Unlike in classical (statistical) mechanics, where a decaying system is treated heuristically and the exponential decay law is easily obtained, the quantum analysis turns out to be involved and sometimes difficult to follow, even for experienced physicists. Scrutiny of the quantum evolution, governed by the Schrödinger equation, unveils the presence of an unavoidable quadratic region at short (sometimes *very* short) times. This region was baptized “Zeno” by Misra and Sudarshan [3] in 1977. The classical allusion to the sophist philosopher is due to an intriguing application: if one frequently interrogates the system, checking whether it is still in its initial state, one can slow down (and eventually stop) its evolution [1, 2, 4]. This is similar to Zeno’s arrow, that would not reach its target if observed at a given position [5].

The purpose of this note is to give an introduction to this topic, addressed to students, young researchers and physicists with no previous knowledge on the subject. This is the summary of a 70-minute lecture [6] delivered in Toruń, Poland, on June 21st, 2012, during the 44th Symposium on Mathematical

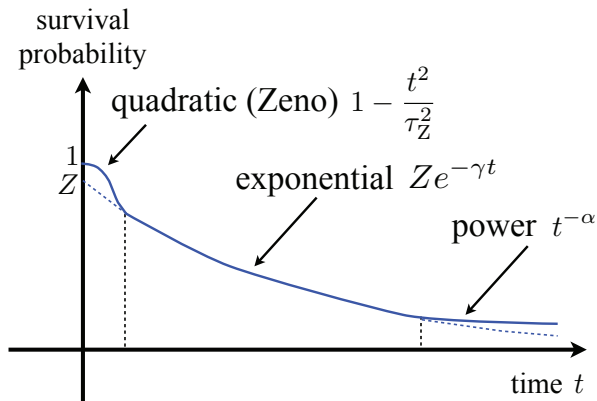


Fig. 1: Survival probability of a decaying quantum system. The initial Zeno region is followed by an exponential decay and finally superseded by a power law. Notice that the extrapolation of the exponential law back to $t = 0$ yields a value Z that is in general $\neq 1$.

Physics on “New Developments in the Theory of Open Quantum Systems”. The audience provided an excellent arena to test the pedagogical aspects of the lecture and helped me understand which facets of the problem are more difficult to grasp. I can only hope that I succeeded in making my presentation as palatable as possible. Some of the examples investigated here have been presented elsewhere [2, 7]. I do not aim at novelty, but rather at clarity, sometimes at the expenses of rigor.

These notes are a somewhat more detailed version of the lecture. The level of the presentation will be kept as elementary as possible. The reader is invited to perform *all* calculations.

In Sect. 2 we review the main features of the quantum evolution law. These are straightforward consequences of the Schrödinger equation. We introduce the quantum Zeno effect in Sect. 3. As anticipated, it is a very general, unavoidable by-product of the quantal dynamics. We then clarify these general aspects by looking at the simplest non-trivial quantum mechanical example (a two-level system) in Sect. 4. We briefly comment on the physical and mathematical origin of the quantum Zeno region in Sect. 5 and on the “meaning” of a von Neumann projective measurement in Sect. 6.

The general analysis of the Zeno effect is disguisingly simple. In Sect. 7 we turn to genuinely unstable systems, that require a quantum field theoretical description, and derive a closed expression for the survival amplitude. The analysis makes use of an analytic continuation in the complex energy plane. Before embarking in this adventure, we remind in Sect. 8 how to perform analytic continuations to the second Riemann sheet, in the presence of a cut

singularity. The analytic continuation of the propagator is done in Sect. 9. We conclude (and apologize) in Sect. 10.

2. The Quantum Mechanical Evolution

2.1. EVOLUTION WITH HERMITIAN HAMILTONIAN

We start off by scrutinizing the quantum-mechanical evolution law, focusing on its short-time features. Let H be the Hamiltonian of a quantum system and $|\psi_0\rangle = |\psi(t=0)\rangle$ its initial state. We shall set henceforth $\hbar = 1$ and assume that all functions to be dealt with are sufficiently regular to admit series expansions. We shall focus on the “survival” amplitude \mathcal{A} and probability p that the system has survived in its initial state $|\psi_0\rangle$ at time t :

$$\mathcal{A}(t) = \langle \psi_0 | \psi(t) \rangle = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle, \quad (1)$$

$$p(t) = |\mathcal{A}(t)|^2 = |\langle \psi_0 | e^{-iHt} | \psi_0 \rangle|^2. \quad (2)$$

Let the system evolve for a short time δt . The Schrödinger equation yields

$$\begin{aligned} |\psi(\delta t)\rangle = e^{-iH\delta t} |\psi_0\rangle &= |\psi_0\rangle - i\langle H \rangle_0 \delta t - \frac{1}{2} H^2 |\psi_0\rangle (\delta t)^2 + O((\delta t)^3) \\ &\equiv |\psi_0\rangle + |\delta\psi\rangle. \end{aligned} \quad (3)$$

The short-time expansion (3) gives

$$\mathcal{A}(\delta t) = 1 - i\langle H \rangle_0 \delta t - \frac{1}{2} \langle H^2 \rangle_0 (\delta t)^2, \quad (4)$$

$$p(\delta t) = 1 - \frac{(\delta t)^2}{\tau_Z^2} + O((\delta t)^4), \quad (5)$$

where $\langle \dots \rangle_0 \equiv \langle \psi_0 | \dots | \psi_0 \rangle$ and

$$\tau_Z^{-2} \equiv \langle H^2 \rangle_0 - \langle H \rangle_0^2, \quad (6)$$

is the Zeno time [2]. In deriving (5) from (4) the Hermiticity of H , ensuring the reality of $\langle H \rangle_0$, played a primary role. Notice that according to (4) the wave function evolves linearly away from the initial state, but the survival probability (of remaining in the initial state) evolves *quadratically* away from 1, due to (5). Recall that due to the unitarity of the evolution, wave functions are always normalised to unity: $\|\psi(t)\| = \|\psi(0)\| = 1$, $\forall t$: the tip of the state vector never leaves the unit sphere. The features of the short time evolution are pictorially displayed in Fig. 2(a).

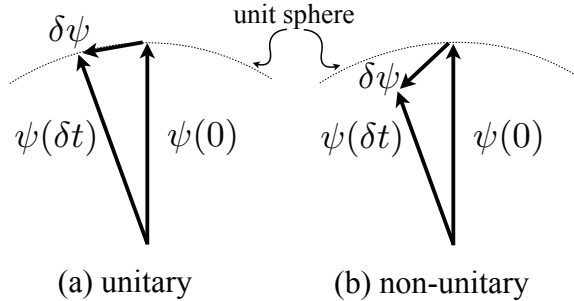


Fig. 2: (a) Unitary evolution engendered by a Hermitian Hamiltonian. The evolution takes place on the unit sphere: $\|\psi(\delta t)\| = \|\psi(0)\| = 1$. (b) Non-unitary evolution engendered by a non-Hermitian Hamiltonian. The tip of the state vector can leave the unit sphere (and enter the unit ball): $\|\psi(\delta t)\| \leq \|\psi(0)\| = 1$. In both cases, $\delta\psi$ is linear in δt .

2.2. EVOLUTION WITH NON-HERMITIAN HAMILTONIAN

Let us add a non-Hermitian part to the Hamiltonian:

$$H' = H - iV, \quad (7)$$

where $V > 0$ is a real “optical” potential (taken to be independent of all dynamical variables — such as position — for simplicity). Optical potentials were frequently used by the founding fathers of nuclear physics, who introduced them in order to describe the coherent scattering of slow neutrons traveling through matter [8].^a

The new survival amplitude and probability read

$$\mathcal{A}'(t) = \langle \psi_0 | \psi(t) \rangle = e^{-Vt} \langle \psi_0 | e^{-iHt} | \psi_0 \rangle, \quad (8)$$

$$p'(t) = e^{-2Vt} |\langle \psi_0 | e^{-iHt} | \psi_0 \rangle|^2. \quad (9)$$

A short-time expansion yields a *linear* behaviour both for amplitude and probability

$$\mathcal{A}'(\delta t) = 1 - (V + i\langle H \rangle_0)\delta t - \frac{1}{2}(\langle H^2 \rangle_0 - V^2 - 2iV\langle H \rangle_0)(\delta t)^2 + O((\delta t)^3), \quad (10)$$

$$p'(\delta t) = 1 - 2V\delta t + O((\delta t)^2). \quad (11)$$

^aThe term “optical” is due to the analogy with the interaction of light with a medium that is both refractive and absorptive. Such an interaction can be analysed by introducing a complex refractive index. Analogously, the scattering and absorption of nucleons by nuclei can be treated by introducing effective neutron-nucleus interaction potentials and by averaging such effective potentials over many nuclei in order to obtain the neutron-matter (complex) optical potential. A consistent expression of V was first derived by Fermi and Zinn [9].

Optical potentials “eat up” probability and account for decay channels. See Fig. 2(b). The tip of the state vector can leave the unit sphere and enter the unit ball: $\|\psi(t)\| \leq \|\psi(0)\| = 1$.^b

In physics, one tends to regard property (5) as more “fundamental”, as it ensues from the Hermiticity of the Hamiltonian and the unitarity of the evolution, that are regarded as very general principles. Yet optical potentials have their own charm and play an important role in effective descriptions of decaying and dissipative systems. Nowadays they have been superseded by the rigorous mathematical framework of Gorini, Kossakowski, Sudarshan and Lindblad [10] that describes the physics of quantum dissipative systems [11, 12, 13].

It is also worth noticing that the exponential law in a quantum context is always the consequence of approximations of some sort. Examples of such approximations can be a macroscopic limit [14] or the intervention of an external apparatus, governed by classical laws, that interacts with the system investigated [15].

2.3. INTERACTION HAMILTONIAN

If the Hamiltonian is composed of a free and an interaction parts

$$H = H_0 + H_{\text{int}} \quad (12)$$

we can obtain an interesting expression, that sheds light on the meaning of the Zeno time. Let $|\psi_n\rangle$ be the eigenstates of the free Hamiltonian, that form a complete set

$$H_0|\psi_n\rangle = \omega_n|\psi_n\rangle. \quad (13)$$

We require that the initial state be an eigenstate of the free Hamiltonian and (as it is customary in quantum field theory) that the interaction be off-diagonal:

$$H_0|\psi_0\rangle = \omega_0|\psi_0\rangle, \quad \langle H_{\text{int}} \rangle_0 = 0. \quad (14)$$

In this interesting case the Zeno time reads

$$\tau_Z^{-2} = \langle H_{\text{int}}^2 \rangle_0 = \sum_n \langle \psi_0 | H_{\text{int}} | \psi_n \rangle \langle \psi_n | H_{\text{int}} | \psi_0 \rangle \quad (15)$$

and depends only on the interaction Hamiltonian.

Formula (15) should be compared to the Fermi “golden rule” [16]^c, yielding the inverse lifetime γ of a decaying quantum system:

$$\gamma = 2\pi \sum_f |\langle \psi_f | H_{\text{int}} | \psi_0 \rangle|^2 \delta(\omega_f - \omega_0), \quad (16)$$

^bIt would leave the unit ball if the optical potential $-iV$ in (7) had the opposite sign.

^cFermi considered expression (16) the *second* golden rule. If you are curious about the first one, see pages 136 and 148 of *Nuclear Physics* [16].

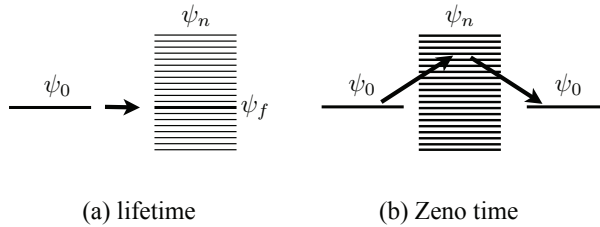


Fig. 3: (a) The lifetime γ in (16) contains only “on-shell” contributions: the delta function entails energy conservation $\omega_f = \omega_0$; ψ_f is in general (very) degenerate (think of an atom in an S -wave emitting a photon: there is a 4π degeneracy in the direction of emission). (b) The Zeno time τ_Z in (15) explores the whole Hilbert space.

where the summation (integral) is over the final states and the continuum limit is implied.

One comment. While (16) contains only “on-shell” contributions (because the delta function ensures energy conservation), the expression (15) explores the *whole* Hilbert space. See Fig. 3.

3. Quantum Zeno Effect

The most familiar formulation of the quantum Zeno effect (QZE) makes use of Von Neumann’s measurements, represented by one-dimensional projectors. Perform N measurements at time intervals $\tau = t/N$, in order to check whether the system is still in its initial state $|\psi_0\rangle$. After each measurement the system’s state is “projected” back onto its initial state $|\psi_0\rangle$ and the evolution starts anew according to Schrödinger’s equation with initial condition $|\psi_0\rangle$. [The system can also be projected onto an orthogonal state $|\psi_0^\perp\rangle$, with (quadratic) probability $1 - p(\tau) = \tau^2/\tau_Z^2$, according to (5). As $\tau = O(1/N)$, such an event becomes increasingly unlikely as N increases.]

The survival probability $p^{(N)}(t)$ at the final time $t = N\tau$ reads

$$\begin{aligned} p^{(N)}(t) &= p(\tau)^N = p(t/N)^N \\ &\simeq [1 - (t/N\tau_Z)^2]^N \stackrel{N \text{ large}}{\longrightarrow} \exp(-t^2/N\tau_Z^2) \stackrel{N \rightarrow \infty}{\longrightarrow} 1, \end{aligned} \quad (17)$$

where we made use of (5). For large N the quantum mechanical evolution is slowed down and in the $N \rightarrow \infty$ limit (infinitely frequent measurements) it is halted, so that the state of the system is “frozen” in its initial state. This is the QZE. It is a consequence of the short-time behaviour (5).

Observe that the survival probability after N pulsed measurements ($t = N\tau$) is interpolated by an exponential law [17]

$$p^{(N)}(t) = p(\tau)^N = \exp(N \log p(\tau)) = \exp(-\gamma_{\text{eff}}(\tau)t), \quad (18)$$

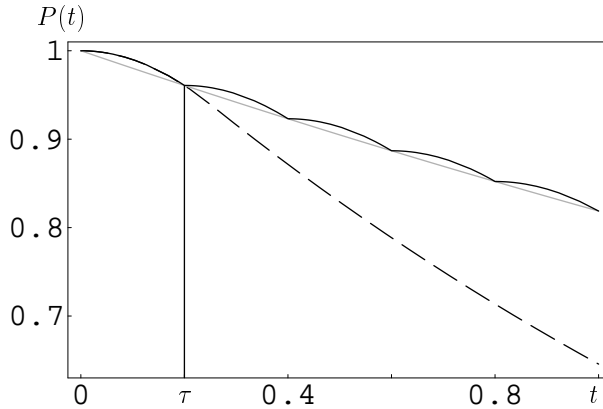


Fig. 4: Quantum Zeno effect for $N = 5$ “pulsed” von Neumann measurements. The dashed (full) line is the survival probability without (with) measurements. The gray line is the interpolating exponential (18). As N increases, $p^{(N)}(t) \rightarrow 1$ uniformly in $[0, t]$. The units on the abscissae are arbitrarily chosen for illustrative purposes.

with an effective decay rate

$$\gamma_{\text{eff}}(\tau) \equiv -\frac{1}{\tau} \log p(\tau). \quad (19)$$

For $\tau \rightarrow 0$ ($N \rightarrow \infty$) one gets from (5) $p(\tau) \simeq \exp(-\tau^2/\tau_Z^2)$, so that

$$\gamma_{\text{eff}}(\tau) \simeq \tau/\tau_Z^2, \quad \tau \rightarrow 0. \quad (20)$$

The Zeno evolution for “pulsed” von Neumann measurements is pictorially represented in Fig. 4.

4. The Simplest Non-trivial Quantum Mechanical Example: The Two-level System

Consider a two-level system undergoing Rabi oscillations. This is the simplest non-trivial quantum mechanical example, for it involves 2×2 matrices and very simple algebra. One can think of an atom shined by a driving laser field whose frequency resonates with one of the atomic transitions, or a neutron spin in a magnetic field. The (interaction) Hamiltonian reads

$$H = H_{\text{int}} = \Omega \sigma_1 = \Omega(|+\rangle\langle-| + |-\rangle\langle+|) = \begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix}, \quad (21)$$

where Ω is a real number, σ_j , $j = 1, 2, 3$, the Pauli matrices and

$$|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (22)$$

are eigenstates of σ_3 . We are neglecting the energy difference between the two states $|\pm\rangle$. Let the initial state be

$$|\psi_0\rangle = |+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (23)$$

so that the evolution yields

$$|\psi(t)\rangle = e^{-iH_{\text{int}}t}|\psi_0\rangle = \cos(\Omega t)|+\rangle - i\sin(\Omega t)|-\rangle = \begin{bmatrix} \cos \Omega t \\ -i \sin \Omega t \end{bmatrix}. \quad (24)$$

The survival amplitude (1) and probability (2) and the Zeno time (6) or (15) read

$$\mathcal{A}(t) = \cos \Omega t, \quad (25)$$

$$p(t) = \cos^2 \Omega t, \quad (26)$$

$$\tau_Z = \Omega^{-1}, \quad (27)$$

respectively. The effective decay rate (19) if N measurements are performed in time t reads

$$\gamma_{\text{eff}}(\tau) = \tau\Omega^2. \quad (28)$$

Look again at Fig. 4.

5. Comments

At the end of the day, the QZE is ascribable to the following mathematical properties of the Schrödinger equation: in a short time $\delta\tau$ ($\sim 1/N$), the phase of the wave function evolves like $O(\delta\tau)$, while the probability changes by $O(\delta\tau^2)$, so that

$$P^{(N)}(t) \simeq [1 - O(1/N^2)]^N \xrightarrow{N \rightarrow \infty} 1. \quad (29)$$

Stated differently, the projection onto the initial state “slowly” evolves away from unity. This is sketched in Fig. 5 and is a very general feature of the Schrödinger equation, as well as of other “fundamental” evolution equations in physics.^d Equations that do not have this feature (e.g. dissipative equations) tend to be regarded as less fundamental, the consequence of approximations of some sort.

^dSuch as the Maxwell equations and (super-)renormalizable quantum field theories.

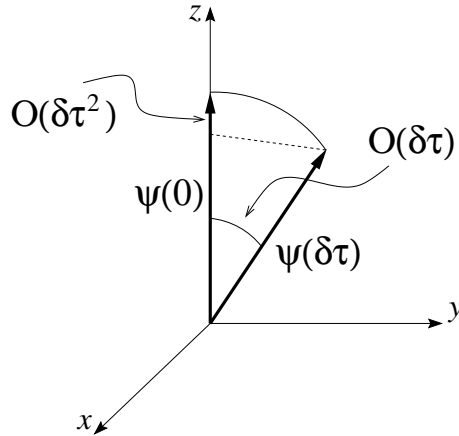


Fig. 5: Short-time evolution of phase and probability: $\delta\tau \sim 1/N$.

6. Unraveling a von Neumann Measurement

What is a (von Neumann [18]) measurement? This is a difficult question, that has been debated for decades and is still a subject of controversy [19]. The mathematical answer is clear, the physical one is not. The evolution due to a measurement process is non-unitary and many reasons lead many physicists (including myself) to think that a von Neumann *projection* is but an effective description of a quantum measurement process. Stated differently, von Neumann’s projectors are a short-hand notation: they summarize the complicated processes that take place in the macroscopic apparatus that perform the measurement and are placed in (macroscopic) regions of space-time [20, 21].

When one deals with the quantum Zeno effect, the situation gets even worse. In general, in order to interact with different “projectors”, the physical system of interest must move, traversing mesoscopic or macroscopic regions of space.^e This dynamics is neglected in most analyses of the QZE: in the elapse of time between two subsequent projections, the system evolves under the action of a Hamiltonian H that does not account for its movement from the region of space where one projection occurs to the (macroscopically) different region of space where the following projection will take place. Think of the example in Sect. 4: everything was neglected but the two-level structure of the system. The physics behind the measurement process is dismissed altogether

^eThere are situations where the system need not move between measurements, but they are rare, and presuppose the existence of a control mechanism that keeps at a given place the physical system undergoing the measurement. An example is an atom in a given position that is shined by a laser: by observing the photons that are scattered/emitted, one can infer which atomic level is populated.

in a single line after (27)! We are so accustomed at computing projections that we do not even think about the underlying physical processes anymore.

We shall henceforth neglect all these problems and act pragmatically. In this section we forget philosophical standpoints and personal taste, and endeavor to give a heuristic description of a quantum measurement, by proposing an effective model for the measuring “apparatus”. Clearly, we are not even hoping of contributing to solving the mystery behind a quantum measurement.

6.1. MIMICKING THE PROJECTION WITH A NON-HERMITIAN HAMILTONIAN

Let us show that the action of a measuring apparatus (performing the von Neumann measurement) can be mimicked by a non-Hermitian Hamiltonian. Consider the Hamiltonian (notation as in Sect. 4)

$$H_{\text{int}} = \begin{bmatrix} 0 & \Omega \\ \Omega & -i2V \end{bmatrix} = -iV \mathbb{1} + \mathbf{h} \cdot \boldsymbol{\sigma}, \quad \mathbf{h} = [\Omega, 0, iV]^T, \quad (30)$$

that yields Rabi oscillations of frequency Ω , but at the same time absorbs away the $|-\rangle$ component of the state vector, performing in this way a “measurement.” H is non-Hermitian, therefore probabilities are not conserved: we are focusing our attention only on the $|+\rangle$ component. State $|-\rangle$ can be viewed as a “decay channel”, according to the discussion in Sect. 2.2.

Elementary algebra [and properties of $\text{SU}(2)$] yields

$$e^{-iH_{\text{int}}t} = e^{-Vt} \left[\cosh(ht) - i \frac{\mathbf{h} \cdot \boldsymbol{\sigma}}{h} \sinh(ht) \right], \quad (31)$$

where $h = \sqrt{V^2 - \Omega^2}$ and we supposed $V \gg \Omega$ (this hypothesis is not vital, but makes the measurement “fast” and therefore effective). The survival amplitude in the initial state (23) reads

$$\begin{aligned} \mathcal{A}(t) &= \langle \psi_0 | e^{-iH_{\text{int}}t} | \psi_0 \rangle \\ &= e^{-Vt} \left[\cosh(ht) + \frac{V}{h} \sinh(ht) \right] \\ &= \frac{1}{2} \left(1 + \frac{V}{h} \right) e^{-(V-h)t} + \frac{1}{2} \left(1 - \frac{V}{h} \right) e^{-(V+h)t}. \end{aligned} \quad (32)$$

Notice the presence of a slow and a fast decay. The survival probability $P(t) = |\mathcal{A}(t)|^2$ is shown in Fig. 6 for $V = 0.4, 2, 10\Omega$.

As expected, probability is (exponentially) absorbed away as $t \rightarrow \infty$. Moreover, for large V , by expanding in the small parameter Ω/V , one finds

$$P(t) \simeq \left(1 + \frac{\Omega^2}{2V} \right) \exp \left(- \frac{\Omega^2}{V} t \right), \quad (33)$$

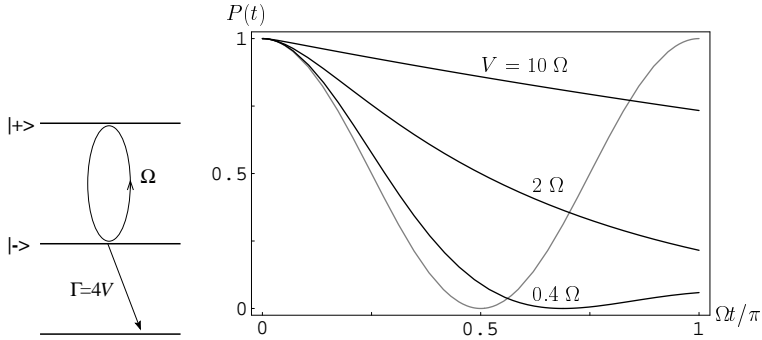


Fig. 6: Survival probability for a system undergoing Rabi oscillations in the presence of absorption ($V = 0.4, 2, 10\Omega$). The gray line is the undisturbed evolution ($V = 0$).

where the wrong normalization at $t = 0$ is an artifact of the approximation (the decay is always quadratic at short times and the above expansion becomes accurate very quickly, on a time scale of order V^{-1}). The effective decay rate $\gamma_{\text{eff}}(V) = \Omega^2/V$ is counterintuitive. Try and show the left panel in Fig. 6 to a friend or a colleague of yours, who has no familiarity with the QZE, and ask the following question: what happens if one initially populates state $|+\rangle$ and increases the decay rate V out of state $|-\rangle$? Chances are that your friend/colleague will reply: state $|+\rangle$ will be depleted faster. Not so: V appears in the *denominator* of the exponent in Eq. (33). Now show your friend the right panel in Fig. 6. The effective lifetime becomes larger as V increases, eventually halting the “decay” (absorption) of the initial state in the $V \rightarrow \infty$ limit. A larger V entails a more “effective” measurement of the initial state. This is an interesting example of QZE.

The global process described here can be viewed as a “continuous” (negative result) measurement performed on the initial state $|+\rangle$. State $|-\rangle$ is continuously monitored with a response time $1/V$: as soon as it becomes populated, it is detected within a time $1/V$. The “strength” V of the observation can be compared to the frequency $\tau^{-1} = (t/N)^{-1}$ of measurements in the “pulsed” formulation of Sect. 3. Indeed, for large values of V one gets from (33)

$$\gamma_{\text{eff}}(V) = \frac{\Omega^2}{V} = \frac{1}{\tau_Z^2 V}, \quad (34)$$

which, compared with (20), yields a cute relation between continuous and pulsed measurements [22]

$$V \simeq 1/\tau. \quad (35)$$

6.2. INTERACTION WITH AN EXTERNAL FIELD YIELDS A NON-HERMITIAN HAMILTONIAN

We now show that the non-Hermitian Hamiltonian (30) can be obtained by considering the evolution engendered by a Hermitian Hamiltonian acting on a larger Hilbert space and then restricting the attention to the subspace spanned by $\{|+\rangle, |-\rangle\}$. Let

$$H = \Omega(|+\rangle\langle-| + |-\rangle\langle+|) + \int d\omega \omega |\omega\rangle\langle\omega| + \sqrt{\frac{\Gamma}{2\pi}} \int d\omega (|-\rangle\langle\omega| + |\omega\rangle\langle-|), \quad (36)$$

that describes a two-level system coupled to a one-dimensional massless boson field in the rotating-wave approximation. Notice that the coupling is “flat”: the two-level system couples to all frequencies in the same way: this enables us to pull out of the last integral a coupling constant $\sqrt{\Gamma}$ that is equal for all frequencies. The state of the system at time t can be written as

$$|\psi(t)\rangle = x(t)|+\rangle + y(t)|-\rangle + \int d\omega z(\omega, t)|\omega\rangle \quad (37)$$

and the Schrödinger equation reads

$$\begin{aligned} i\dot{x}(t) &= \Omega y(t), \\ iy(t) &= \Omega x(t) + \sqrt{\frac{\Gamma}{2\pi}} \int d\omega z(\omega, t), \\ iz(\omega, t) &= \omega z(\omega, t) + \sqrt{\frac{\Gamma}{2\pi}} y(t). \end{aligned} \quad (38)$$

By using the initial condition $x(0) = 1$ and $y(0) = z(\omega, 0) = 0$ one obtains

$$z(\omega, t) = -i\sqrt{\frac{\Gamma}{2\pi}} \int_0^t d\tau e^{-i\omega(t-\tau)} y(\tau) \quad (39)$$

and

$$iy(t) = \Omega x(t) - i\frac{\Gamma}{2\pi} \int d\omega \int_0^t d\tau e^{-i\omega(t-\tau)} y(\tau) = \Omega x(t) - i\frac{\Gamma}{2} y(t). \quad (40)$$

Observe that in order to obtain this result the integral over ω has to be extended over the whole real line (from $-\infty$ to $+\infty$). Also, $\int_0^t \delta(t-\tau) d\tau = 1/2$.

The only remnant of the coupling of the qubit to the continuum of levels is the appearance of the imaginary frequency $-i\Gamma/2$. This is ascribable

to the afore-mentioned “flatness” of the coupling [there is no form factor or frequency cutoff in the interaction term of (36)], which yields a purely exponential (Markovian) decay of $y(t)$.

In conclusion, $z(\omega, t)$ drops out of the first two equations (38), that now describe the (reduced) dynamics in the subspace spanned by $|+\rangle$ and $|-\rangle$:

$$\begin{aligned} i\dot{x}(t) &= \Omega y(t), \\ i\dot{y}(t) &= -i\frac{\Gamma}{2}y + \Omega x(t). \end{aligned} \quad (41)$$

Of course, this dynamics is not unitary, for probability flows out of the subspace, and is generated by the non-Hermitian Hamiltonian

$$H = \Omega(|+\rangle\langle-| + |-\rangle\langle+|) - i\frac{\Gamma}{2}|-\rangle\langle-| = \begin{bmatrix} 0 & \Omega \\ \Omega & -i\Gamma/2 \end{bmatrix}. \quad (42)$$

This Hamiltonian is the same as (30) when one sets $\Gamma = 4V$. QZE is obtained by increasing Γ : a larger coupling to the environment leads to a more effective “continuous” observation on the system (quicker response of the measuring apparatus), and as a consequence to slower decay (QZE). Try and ask the same tricky question mentioned after (33) to another friend/colleague. Rather than the left panel in Fig. 6, draw a figure in which level $|-\rangle$ decays to a photon field, and increase the coupling Γ between them.

We leave it to the reader to judge whether the analysis of the measurement process proposed in this section is more satisfactory than that outlined in Sect. 6.1. We generally tend to regard this description more “complete” than that proposed in Sect. 6.1. One should notice that in this section quantum field theory has sneaked into the picture: equation (36) describes a quantum field.

7. Genuine Unstable Systems and Field Theory

We shall now forget about quantum measurements and QZE and focus on the non-exponential features of decay. The arguments given in Sect. 2.1 are very general and cannot be rejected: decay cannot be exponential at short times. However, it is of great interest to discuss this problem in a quantum field theoretical framework. This will help us focus on the important role played by the form factors of the interaction.

We start by generalizing the two-level Hamiltonian (21) to N states $|j\rangle$ ($j = 1, \dots, N$) with different energies

$$H_0 = \omega_0|+\rangle\langle+| + \sum_{j=1}^N \omega_j|j\rangle\langle j| = \begin{bmatrix} \omega_0 & 0 & \dots & 0 \\ 0 & \omega_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_N \end{bmatrix} \quad (43)$$

and (real) couplings

$$H_{\text{int}} = \sum_{j=1}^N g_j (|+\rangle\langle j| + |j\rangle\langle +|) = \begin{bmatrix} 0 & g_1 & \dots & g_N \\ g_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & 0 & \dots & 0 \end{bmatrix}. \quad (44)$$

In order to obtain a truly unstable system we need a continuous spectrum, so we consider the continuum limit $\omega_j \rightarrow \omega$, $|j\rangle \rightarrow \sqrt{\delta\omega}|\omega\rangle$, $g_j \rightarrow \sqrt{\delta\omega}g(\omega)$, with $\delta\omega \rightarrow 0$,

$$H = H_0 + H_{\text{int}} = \omega_0 |+\rangle\langle +| + \int d\omega \omega |\omega\rangle\langle \omega| + \int d\omega g(\omega) (|+\rangle\langle \omega| + |\omega\rangle\langle +|). \quad (45)$$

State $|+\rangle$ is normalizable, but states $|\omega\rangle$ are not:

$$\langle +|+\rangle = 1, \quad \langle \omega|\omega'\rangle = \delta(\omega - \omega'), \quad \langle +|\omega\rangle = 0. \quad (46)$$

$\{|+\rangle, |\omega\rangle\}$ is the eigenbasis of H_0 and is a resolution of the identity

$$|+\rangle\langle +| + \int d\omega |\omega\rangle\langle \omega| = 1. \quad (47)$$

As before, we take as initial state $|\psi_0\rangle = |+\rangle$. The interaction of this state with the continuum of states $|\omega\rangle$ is responsible for its decay and depends on the *form factor* $g(\omega)$. We assumed (with no loss of generality) $g(\omega)$ to be real.

It is worth stressing that the purpose of studying model (45) is very different from the motivations that led us to analyse model (36). In Sect. 6.2 we were interested in the QZE on level $|+\rangle$ that arises when level $|-\rangle$ is “measured”, while in this section we focus on the deviations from exponential when level $|+\rangle$ is coupled to a continuum. There is no level $|-\rangle$ here.^f

The Fourier-Laplace transform of the survival amplitude (1) for this model can be given in a convenient analytic expression. The transform of the survival amplitude is the expectation value of the *resolvent*

$$\mathcal{A}(E) = \int_0^\infty dt e^{iEt} \mathcal{A}(t) = \langle +| \int_0^\infty dt e^{iEt} e^{-iHt} |+\rangle = \langle +| \frac{i}{E - H} |+\rangle \quad (48)$$

and is defined for $\text{Im } E > 0$ (check!). By using twice the operator identity

$$\frac{1}{E - H} = \frac{1}{E - H_0} + \frac{1}{E - H_0} H_{\text{int}} \frac{1}{E - H} \quad (49)$$

^fAlthough it would not be difficult to introduce it.

one obtains

$$\begin{aligned}
 \mathcal{A}(E) &= \langle + | \left[\frac{i}{E - H_0} + \frac{1}{E - H_0} H_{\text{int}} \frac{i}{E - H_0} \right. \\
 &\quad \left. + \frac{1}{E - H_0} H_{\text{int}} \frac{1}{E - H_0} H_{\text{int}} \frac{i}{E - H} \right] | + \rangle \\
 &= \frac{i}{E - \omega_0} + \frac{1}{E - \omega_0} \int d\omega \frac{|\langle + | H_{\text{int}} | \omega \rangle|^2}{E - \omega} \mathcal{A}(E). \quad (50)
 \end{aligned}$$

In the above derivation we used the resolution (47) of the identity and the fact that H_{int} is completely off-diagonal in the eigenbasis of H_0 , compare (14). The advantage of looking at the Fourier-Laplace transform (48) lies in the fact that (50) is algebraic and can be solved to yield

$$\mathcal{A}(E) = \frac{i}{E - \omega_0 - \Sigma(E)}, \quad (51)$$

where the *self-energy function* $\Sigma(E)$ is related to the form factor $g(\omega)$ by a simple integration

$$\Sigma(E) = \int d\omega \frac{|\langle + | H_{\text{int}} | \omega \rangle|^2}{E - \omega} = \int d\omega \frac{g^2(\omega)}{E - \omega}. \quad (52)$$

Notice that the self-energy function is a “small” quantity, being proportional to the square of the coupling between level $|+\rangle$ and the continuum. By inverting (48) we finally get

$$\mathcal{A}(t) = \int_{\text{B}} \frac{dE}{2\pi} e^{-iEt} \mathcal{A}(E) = \frac{i}{2\pi} \int_{\text{B}} dE \frac{e^{-iEt}}{E - \omega_0 - \Sigma(E)}, \quad (53)$$

the Bromwich path B being a horizontal line $\text{Im } E = \text{constant} > 0$ in the half plane of convergence of the Fourier-Laplace transform (upper half plane). This is the quantity we sought, expressed in terms of a quadrature.

So far, the analysis is general and valid for any state. We shall now consider the case of an unstable system. However, before doing so, we shall give some mathematical notions related to complex analysis and analytic continuation.

8. Intermezzo: Analytic Continuation on the Second Riemann Sheet

Consider the function

$$F(z) = \int_0^{\infty} dE \frac{f(E)}{E - z}, \quad (54)$$

where $z = x + iy \in \mathbb{C}$, f is a smooth function and E a real variable. F is an analytic function in the complex z plane, but has a (logarithmic) cut for positive real z . We obtain

$$\begin{aligned} F(x \pm i0^+) &= \int_0^{\infty} dE \frac{f(E)}{E - x \mp i0^+} \\ &= \int_0^{\infty} dE f(E) \left(\frac{P}{E - x} \pm i\pi\delta(E - x) \right), \end{aligned} \quad (55)$$

where P denotes principal value. The discontinuity across the cut is therefore

$$\begin{aligned} F(x + i0^+) - F(x - i0^+) &= 2\pi i \int_0^{\infty} dE f(E) \delta(E - x) = 2\pi i f(x) \\ &\quad (x > 0). \end{aligned} \quad (56)$$

Clearly, in (56) the function $F(x \pm i0^+)$ is evaluated on the *first* Riemann sheet, immediately above and below the cut on the positive real half-line.

Let's now smoothly cross the positive real half-line, going from the first to the second Riemann sheet. The value of $F(z)$ above the real axis, on the first sheet, and below it, on the second sheet, is the same by definition:

$$F(x + i0^+) = F_{\text{II}}(x - i0^+) \quad (x > 0), \quad (57)$$

where F_{II} is the function evaluated on the second Riemann sheet. By using (56)–(57), one gets

$$F_{\text{II}}(x - i0^+) = F(x - i0^+) + 2\pi i f(x) \quad (x > 0). \quad (58)$$

Therefore the “jump” (56) of $F(z)$ evaluated on the two edges of the cut (on the first Riemann sheet) is equal to the difference of the values of the function evaluated on the second and first sheet. By analytically extending formula (58) one obtains

$$F_{\text{II}}(z) = F(z) + 2\pi i f(z), \quad \forall z \in \mathbb{C}. \quad (59)$$

It is obvious that in the above considerations we are implicitly assuming that analytic continuation is licit. Assume now that $F(z)$ in (54) be defined for $\text{Im } z > 0$ and we want to extend it to the region $\text{Im } z < 0$. It is easy to see that the definition

$$F(z) = \int_{\Gamma} dE \frac{f(E)}{E - z}, \quad \text{for } \text{Im } z > 0 \quad (60)$$

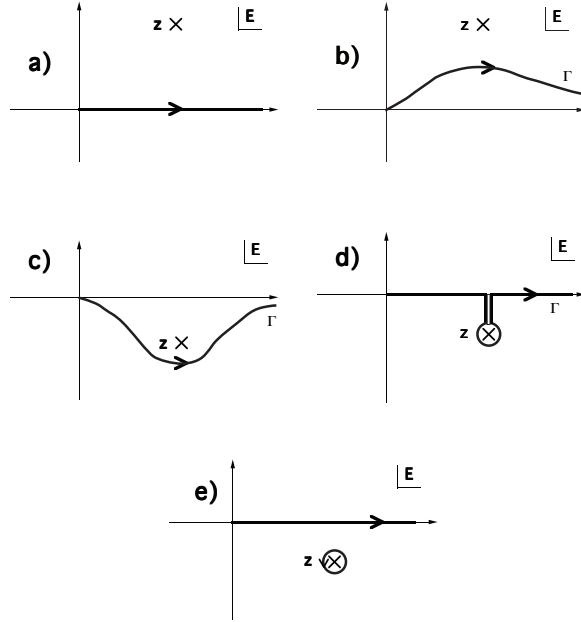


Fig. 7: Analytic continuation across the cut in the complex E -plane. a) equation (54); b) equation (60); c)-e) equation (61).

is equivalent to (54), as far as the contour Γ starts at the origin and reaches $+\infty$ by remaining below z . Notice that E in (60) takes complex values and Γ can be arbitrarily deformed, as far as its configuration with respect to the singularity z is respected. See Fig. 7b.

The extension to the case $\text{Im } z < 0$ is straightforward: when z smoothly crosses the positive real axis, going to the second Riemann sheet, the contour integration in the complex E plane remains below z , respecting the position of the singularity. This yields again the result (59): the contour is first deformed in order to remain below z , then deformed into a small circle, that runs counterclockwise around z , plus the original contour

$$\begin{aligned}
 F(x + iy) \xrightarrow{y < 0} F_{\text{II}}(x + iy) &= \int_{\Gamma} dE \frac{f(E)}{E - x - iy} \\
 &= \int_0^{\infty} dE \frac{f(E)}{E - x - iy} + 2\pi i f(x + iy) \\
 &= F(x + iy) + 2\pi i f(x + iy). \quad (61)
 \end{aligned}$$

This is identical to (59). In this case the difference between F and F_{II} is given by the pole. See Fig. 7c-e. These beautiful mathematical ideas will be

very useful to analyse the behaviour of the propagator (53).

9. Analytic Continuation of the Propagator

The function $\mathcal{A}(E)$ in (51), (53) has a branching point at $E = \omega_g$, the lower bound of the continuous spectrum of the Hamiltonian H , a cut that extends to $E = +\infty$ and no additional singularities on the first Riemann sheet, while singularities can appear on the second sheet. These important features were studied by Araki *et al.* [23] and Schwinger [24] in the 50's.[§]

Indeed, $\mathcal{A}(E)$ is defined for $\text{Im } E > 0$, so that its Fourier transform, the survival amplitude (53), converges for $t > 0$. When the self-energy function is analytically continued to the second Riemann sheet, the contour must be modified so that its position with respect to the singularity is maintained.

The initial state has energy $\omega_0 > \omega_g$ and is therefore embedded in the continuous spectrum of H . If $|\Sigma(\omega_g)| < \omega_0$ (which happens for sufficiently smooth form factors and small coupling), the resolvent is analytic in the whole complex plane cut along the real axis (continuous spectrum of H) [23, 24]. On the other hand, there exists a pole E_{pole} located just below the branch cut in the second Riemann sheet, solution of the equation

$$E_{\text{pole}} - \omega_0 - \Sigma_{\text{II}}(E_{\text{pole}}) = 0, \quad (62)$$

Σ_{II} being the determination of the self-energy function in the second sheet. Remember that the self-energy function is a “small” quantity, being proportional to the square of the coupling between level $|+\rangle$ and the continuum: the pole E_{pole} is therefore very close to ω_0 . See Fig. 8.

The pole has a real and imaginary part

$$E_{\text{pole}} = \omega_0 + \delta\omega_0 - i\gamma/2, \quad (63)$$

that can be easily computed by following the mathematical technique outlined in the previous section

$$\delta\omega_0 = \text{Re } \Sigma_{\text{II}}(E_{\text{pole}}) \simeq \text{Re } \Sigma(\omega_0 + i0^+) = \text{P} \int d\omega \frac{g^2(\omega)}{\omega_0 - \omega}, \quad (64)$$

$$\gamma = -2 \text{Im } \Sigma_{\text{II}}(E_{\text{pole}}) \simeq -2 \text{Im } \Sigma(\omega_0 + i0^+) = 2\pi g^2(\omega_0). \quad (65)$$

In the above formulas, $\delta\omega_0$ is the energy shift and γ the inverse lifetime, according to the Fermi “golden” rule [16]. Both quantities are written at

[§]Those were the golden years of renormalization in quantum field theory.

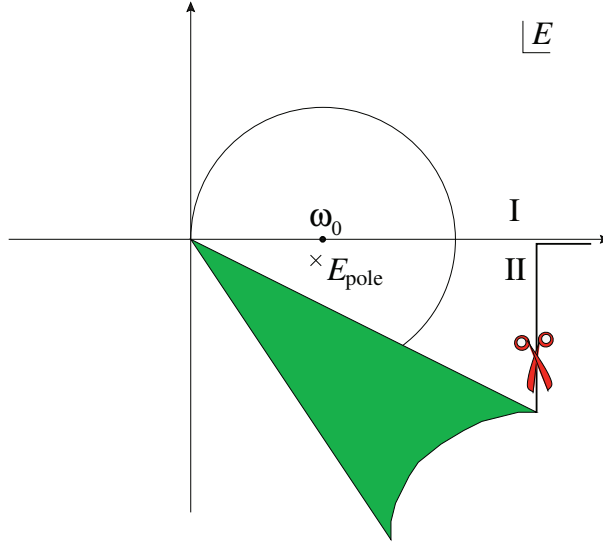


Fig. 8: The pole E_{pole} on the second Riemann sheet is (coupling constant)²-close to ω_0 : see (53) and (62). We drew the circle of convergence of an asymptotic expansion around ω_0 . The derivation of (51) from (48) requires the definition of the self-energy function (52). Try and understand which mathematical hypotheses are needed.

second order in the coupling constant. Check that γ is the same quantity that appears in (16).^h

In conclusion, the survival amplitude (53) has the general form

$$\mathcal{A}(t) = \mathcal{A}_{\text{pole}}(t) + \mathcal{A}_{\text{cut}}(t), \quad (66)$$

where

$$\mathcal{A}_{\text{pole}}(t) = \frac{e^{-i(\omega_0 + \delta\omega_0)t - \gamma t/2}}{1 - \Sigma'_{\text{II}}(E_{\text{pole}})}, \quad (67)$$

is due to the pole contribution (62) and

$$\mathcal{A}_{\text{cut}}(t) = \frac{i}{2\pi} \int_{\text{cut}} dE \frac{e^{-iEt}}{E - \omega_0 - \Sigma(E)}, \quad (68)$$

^hThe derivation of (64)–(65) is left as an exercise (a very useful one). Be careful in deriving γ in (65), you might miss a factor 2. Modern literature (unlike classic literature) is plagued by missing factors 2. The correct solution is obtained by using the formula

$$\lim_{\gamma \rightarrow 0} \frac{\gamma}{E^2 + \frac{\gamma^2}{4}} = 2\pi\delta(E),$$

that is valid because γ is a small quantity (second order in the coupling constant), and neglecting fourth-order terms in the coupling constant.

is the branch-cut contribution, as explained in the previous section: see Fig. 7e).

It is not difficult to see that, if the coupling is small, at intermediate times the pole contribution dominates the evolution and

$$P(t) \simeq |\mathcal{A}_{\text{pole}}(t)|^2 = Z e^{-\gamma t}, \quad Z = |1 - \Sigma'_{\text{II}}(E_{\text{pole}})|^{-2}, \quad (69)$$

where Z , the intersection of the asymptotic exponential with the $t = 0$ axis, is the so-called wave-function renormalization. This explains the behaviour sketched in Fig. 1. It would be interesting to see [1] that the cut contribution (68) *cannot* be neglected at short and long times, where it yields the quadratic Zeno behaviour and the power tail, respectively.

9.1. A FEW OBSERVATIONS

In order to obtain a purely exponential decay, one can simply neglect the branch cut contribution altogether and retain only the dominant contribution of the pole singularity. An interesting way to obtain the desired result is to replace the self-energy function with a constant (equal to its value at the pole) in (51):

$$\mathcal{A}(E) \longrightarrow \frac{i}{E - \omega_0 - \Sigma_{\text{II}}(E_{\text{pole}})} = \frac{i}{E - E_{\text{pole}}} \equiv \mathcal{A}^{\text{W}^2}(E), \quad (70)$$

where we used the pole equation (62) in the central equality. This is the celebrated Weisskopf-Wigner approximation [25] and yields a purely exponential behaviour, $\mathcal{A}(t) = \exp(-iE_{\text{pole}}t)$, without short- and long-time corrections.ⁱ

Another nice way to obtain a purely exponential decay is to replace the form factor g in (45) by a constant value, say $\sqrt{\gamma/2\pi}$. This is a useful exercise. (Hint: follow the same strategy as in Sect. 6.2.)

Another important problem is the duration of the non-exponential Zeno region and the onset to the power law. The answer to these questions requires careful evaluation of the cut contribution (68). One finds that the Zeno region is superseded by the exponential decay after a time of the order of the inverse frequency cutoff of the form factor g in the interaction Hamiltonian (45) and the exponential is superseded by a power law after a time of the order of a significant number (say 10^2) of lifetimes. However, these conclusions are model-dependent and neglect important numerical factors. As a general rule, time evolutions in quantum field theory are a complex problem [26] and lead to the inverse Zeno effect [27, 28, 17].

ⁱMy former teacher M. Namiki used to tell me that great physicists know in advance the result they want to get and use mathematics in a “creative” way to obtain what they need. The older I get, the more I agree.

10. Conclusions and Apologies

The title of these notes is “All you ever wanted to know about the quantum Zeno effect in 70 minutes”. Admittedly, I lied: my lecture would have lasted 90 minutes, if my chairman had not (very politely) stopped me. However, the title contains a second, more deceitful lie: these notes are by no means *all* you ever wanted to know about the QZE. However, I do not feel guilty about the second (white) lie. The main purpose of a lecture is not to explain everything; rather, it is to make students curious, so that they can go and deepen the subject. This contains, in embryo, what we nowadays call curiosity-driven research. If I managed to get my students interested, my lecture was successful.

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