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Deviations from exponential law and Van Hove's “ $\lambda^2 t$ ” limit

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Abstract

The deviations from a purely exponential behavior in a decay process are analyzed in relation to Van Hove's “ $\lambda^2 t$ ” limiting procedure. Our attention is focused on the effects that arise when the coupling constant is small but nonvanishing. We first consider a simple model (two-level atom in interaction with the electromagnetic field), then gradually extend our analysis to a more general framework. We estimate all deviations from exponential behavior at leading orders in the coupling constant. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The evolution law in quantum mechanics is governed by unitary operators [1]. This entails, by virtue of very general mathematical properties, that the decay of an unstable quantum system cannot be purely exponential. In general, a rigorous analysis based on the Schrödinger equation shows that the decay law is quadratic for very short times [2–8] and governed by a power law for very long times [9–14]. These features of the quantum evolution are well known and discussed in textbooks of quantum mechanics [15,16] and quantum field theory [17]. The temporal behavior of quantum systems is reviewed in Ref. [18].

Although the domain of validity of the exponential law is limited, the Fermi “Golden Rule” [19–21] works very well and no deviations from the exponential behavior have

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ever been observed for truly unstable systems [22,23]. Although non-exponential leakage through a potential barrier has recently been detected: see [24]. The quantum mechanical derivation of this law is based on the sensible idea that the temporal evolution of a quantum system is dominated by a pole near the real axis of the complex energy plane (Weisskopf–Wigner approximation [25–28]). This yields an irreversible evolution, characterized by a master equation and exponential decay [29,30]. An important contribution to this issue was given in the 1950s by Van Hove [31], who rigorously showed that it is possible to obtain a master equation (leading to exponential behavior) for a quantum mechanical system endowed with many (infinite) degrees of freedom, by making use of the so-called “ $\lambda^2 t$ ” limit. The crucial idea is to consider the limit

$$\lambda \rightarrow 0 \quad \text{keeping} \quad \tilde{t} = \lambda^2 t \quad \text{finite} \quad (\lambda\text{-independent constant}), \quad (1.1)$$

where λ is the coupling constant and t time. One then looks at the evolution of the quantum system as a function of the rescaled time \tilde{t} . There has recently been a renewed interest in the physical literature for this time-scale transformation and its subtle mathematical features: see [32,33].

The purpose of this paper is to consider the effects that arise when the coupling constant is small but nonvanishing. This will enable us to give general estimates for deviations from exponential behavior. The paper is organized as follows. We shall first look, in Section 2, at a simple system: we summarize some recent results on a characteristic transition of the hydrogen atom in the two-level approximation. In Section 3 we consider the action of the Van Hove limiting procedure on a generic two-level atom in the rotating-wave approximation and then generalize our result when the other discrete levels and the counter-rotating terms are taken into account. We look in particular at the scaling procedure from the perspective of the complex energy plane, rather than in terms of the time variable. This enables us to pin down the different sources of nonexponential behavior. In Section 4 our analysis is extended to a general field-theoretical framework: general estimates are given of all deviations from the exponential law (both at short and long times) at leading orders in the coupling constant.

2. Hydrogen atom in the two-level approximation

We start our considerations from a simple field-theoretical model. Consider the Hamiltonian ($\hbar = c = 1$)

$$H = H_0 + \lambda V, \quad (2.1)$$

$$H_0 = H_{\text{atom}} + H_{\text{EM}} \\ \equiv \omega_0 b_2^\dagger b_2 + \sum_{\beta} \int_0^{\infty} d\omega \omega a_{\omega\beta}^\dagger a_{\omega\beta}, \quad (2.2)$$

$$V = \sum_{\beta} \int_0^{\infty} d\omega [\varphi_{\beta}(\omega) b_1^\dagger b_2 a_{\omega\beta}^\dagger + \varphi_{\beta}^*(\omega) b_2^\dagger b_1 a_{\omega\beta}], \quad (2.3)$$

where H_{atom} is the free Hamiltonian of a two-level atom (ω_0 being the energy gap between the two atomic levels), b_j, b_j^\dagger are the annihilation and creation operators of the atomic level j , obeying anticommutation relations

$$\{b_k, b_\ell^\dagger\} = \delta_{k\ell} \quad (k, \ell = 1, 2), \tag{2.4}$$

H_{EM} is the Hamiltonian of the free EM field, λ the coupling constant and V the interaction Hamiltonian. We are working in the rotating-wave approximation and with the energy-angular momentum basis for photons [34,35], with $\sum_\beta = \sum_{j=1}^\infty \sum_{m=-j}^j \sum_{\varepsilon=0}^1$, where ε defines the photon parity $P = (-1)^{j+1+\varepsilon}$, j is the total angular momentum (orbital+spin) of the photon, m its magnetic quantum number and

$$[a_{\omega j m \varepsilon}, a_{\omega' j' m' \varepsilon'}^\dagger] = \delta(\omega - \omega') \delta_{jj'} \delta_{mm'} \delta_{\varepsilon\varepsilon'}. \tag{2.5}$$

We shall focus our attention on the 2P–1S transition of hydrogen, so that $\omega_0 = \frac{3}{8} \alpha^2 m_e \simeq 1.550 \times 10^{16}$ rad/s (α is the fine structure constant and m_e the electron mass) and the matrix elements $\varphi_\beta(\omega)$ of the interaction are known exactly [36–38]:

$$\begin{aligned} \varphi_\beta(\omega) &= \langle 1; 1_{\omega\beta} | V | 2; 0 \rangle = \varphi_{\bar{\beta}}(\omega) \delta_{\beta\bar{\beta}} \\ &= i(A)^{1/2} \frac{(\omega/A)^{1/2}}{1 + (\omega/A)^2} \delta_{j1} \delta_{m m_2} \delta_{\varepsilon 1}, \end{aligned} \tag{2.6}$$

where

$$|1; 1_{\omega\beta}\rangle \equiv |1\rangle \otimes |\omega, j, m, \varepsilon\rangle, \quad |2; 0\rangle \equiv |2\rangle \otimes |0\rangle \tag{2.7}$$

(the first ket refers to the atom and the second to the photon) and the selection rule, due to angular momentum and parity conservation, entails that the only nonvanishing term in (2.3) and (2.6) is $\bar{\beta} = (1, m_2, 1)$. We emphasize that the so-called “retardation effects” are taken into account in (2.6). The normalization reads

$$\langle 1; 1_{\omega\beta} | 1; 1_{\omega'\beta'} \rangle = \delta(\omega - \omega') \delta_{\beta\beta'}, \quad \langle 2; 0 | 2; 0 \rangle = 1 \tag{2.8}$$

and the quantities

$$\begin{aligned} A &= \frac{3}{2} \alpha m_e = \frac{3}{2a_0} \simeq 8.498 \times 10^{18} \text{ rad/s}, \\ \lambda &= \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{2}{3}\right)^{9/2} \alpha^{3/2} \simeq 0.802 \times 10^{-4} \end{aligned} \tag{2.9}$$

are the natural cutoff of the atomic form factor, expressed in terms of the Bohr radius a_0 , and the coupling constant, respectively. Observe that there are no free parameters in (2.1)–(2.3).

The above model was analyzed in a previous paper [39], where we mainly concentrated our attention on the deviations from exponential, both at short and long times. There is interesting related work on this subject [40–45]. Let us summarize the main results, by concentrating our attention on the role played by the coupling constant λ .

Assume one can prepare (at time $t=0$) the atom in the initial state $|2; 0\rangle$. This is an eigenstate of the unperturbed Hamiltonian H_0 , whose eigenvalue is ω_0 . The evolution

is governed by the unitary operator $U(t) = \exp(-iHt)$ and the “survival” or nondecay amplitude and probability at time t are defined as (interaction picture)

$$\mathcal{A}(t) = \langle 2; 0 | e^{iH_0 t} U(t) | 2; 0 \rangle, \tag{2.10}$$

$$P(t) = |\langle 2; 0 | e^{iH_0 t} U(t) | 2; 0 \rangle|^2. \tag{2.11}$$

The survival probability at short times reads

$$P(t) = 1 - t^2/\tau_Z^2 + \dots, \quad \tau_Z \equiv (\lambda^2 \langle 2; 0 | V^2 | 2; 0 \rangle)^{-1/2}. \tag{2.12}$$

The quantity τ_Z is the so-called “Zeno time” and yields a quantitative estimate of the deviation from exponential at very short times. Strictly speaking, τ_Z is the convexity of $P(t)$ in the origin. One finds [39]

$$\tau_Z = \frac{\sqrt{6}}{\lambda A} = (3\pi)^{1/2} \left(\frac{3}{2}\right)^{7/2} \frac{1}{\alpha^{5/2} m_e} \simeq 3.593 \times 10^{-15} \text{ s}. \tag{2.13}$$

It is possible to obtain a closed expression for $\mathcal{A}(t)$, valid at all times, as an inverse Laplace transform:

$$\mathcal{A}(t) = \frac{e^{i\omega_0 t}}{2\pi i} \int_B ds \frac{e^{sAt}}{s + i\omega_0/A + \lambda^2 Q(s)}, \tag{2.14}$$

$$Q(s) \equiv -i \int_0^\infty dx \frac{x}{(1+x^2)^4} \frac{1}{x - is}, \tag{2.15}$$

where B is the Bromwich path, i.e. a vertical line at the right of all the singularities of the integrand. Q is a self-energy contribution and can be computed exactly:

$$Q(s) = \frac{-15\pi i - (88 - 48\pi i)s - 45\pi i s^2 + 144s^3}{96(s^2 - 1)^4} + \frac{15\pi i s^4 - 72s^5 - 3\pi i s^6 + 16s^7 - 96s \log s}{96(s^2 - 1)^4}. \tag{2.16}$$

At short and long times one gets

$$P(t) \sim 1 - \frac{t^2}{\tau_Z^2} \quad (t \ll \tau_Z), \tag{2.17}$$

$$P(t) \sim \mathcal{Z}^2 e^{-\gamma t} + \lambda^4 \frac{\mathcal{C}^2}{(\omega_0 t)^4} - 2\lambda^2 \frac{\mathcal{C}\mathcal{Z}}{(\omega_0 t)^2} e^{-\gamma/2t} \cos[(\omega_0 - \Delta E)t - \zeta] \quad (t \gg A^{-1}), \tag{2.18}$$

where

$$\gamma = 2\pi\lambda^2 |\varphi_{\bar{\beta}}(\omega_0)|^2 + O(\lambda^4) = 2\pi\lambda^2 \omega_0 + O(\lambda^4) \simeq 6.268 \times 10^8 \text{ s}^{-1}, \tag{2.19}$$

$$\Delta E = \lambda^2 \mathcal{P} \int_0^\infty d\omega |\varphi_{\bar{\beta}}(\omega)|^2 \frac{1}{\omega - \omega_0} + O(\lambda^4) \simeq 0.5\lambda^2 A, \tag{2.20}$$

$$\mathcal{Z} e^{i\zeta} \simeq (1 - 4.38\lambda^2) e^{-i1.00\pi\lambda^2} = 1 + O(\lambda^2), \tag{2.21}$$

$$\mathcal{C} \simeq 1 + 5.38\lambda^2 = 1 + O(\lambda^2). \tag{2.22}$$

The first two formulae give the Fermi “Golden Rule”, yielding the lifetime

$$\tau_E = \gamma^{-1} \simeq 1.595 \times 10^{-9} \text{ s} \tag{2.23}$$

and the second-order correction to the energy level ω_0 . The exact expressions for quantities (2.19)–(2.22) are given in Ref. [39].

3. Van Hove’s limit

Let us look at Van Hove’s “ $\lambda^2 t$ ” limiting procedure applied to the model of the previous section. Before proceeding to a detailed analysis, it is worth putting forward a few preliminary remarks: we shall scrutinize (in terms of the coupling constant) the mechanisms that make the nonexponential contributions in (2.17)–(2.18) vanish. To this end, observe first that as $\lambda \rightarrow 0$ the Zeno time (2.13) diverges, while the *rescaled* Zeno time vanishes:

$$\tilde{\tau}_Z \equiv \lambda^2 \tau_Z = \lambda \frac{\sqrt{6}}{A} = O(\lambda). \tag{3.1}$$

On the other hand, the rescaled lifetime (2.23) remains constant [see (2.19)]:

$$\tilde{\tau}_E \equiv \lambda^2 \tau_E = \frac{1}{2\pi\omega_0} = O(1). \tag{3.2}$$

Moreover, the transition to a power law occurs when the first two terms in the right-hand side of (2.18) are comparable, so that

$$(\omega_0 t)^2 e^{-\gamma/2t} \simeq \lambda^2, \tag{3.3}$$

because both \mathcal{C} and \mathcal{Z} are $\simeq 1$. In the limit of small λ , (3.3) yields $t = \tau_{\text{pow}}$, with

$$2 \log(\omega_0 \tau_{\text{pow}}) - \frac{\gamma}{2} \tau_{\text{pow}} \simeq 2 \log \lambda, \tag{3.4}$$

namely, by (2.19),

$$\frac{\tau_{\text{pow}}}{\tau_E} \simeq 4 \log \frac{1}{\lambda} + 4 \log \frac{\tau_{\text{pow}}}{2\pi\lambda^2\tau_E} = 12 \log \frac{1}{\lambda} + 4 \log \frac{\tau_{\text{pow}}}{\tau_E} + 4 \log \frac{1}{2\pi}. \tag{3.5}$$

Therefore, when time is rescaled,

$$\tilde{\tau}_{\text{pow}} \equiv \lambda^2 \tau_{\text{pow}} = 12 \tilde{\tau}_E \log \frac{1}{\lambda} + O\left(\log \log \frac{1}{\lambda}\right) = O\left(\log \frac{1}{\lambda}\right). \tag{3.6}$$

Finally, the power contributions are $\sim O(\lambda^{3\alpha}) t^{-\alpha}$ ($\alpha=2, 4$), the period of the oscillations [last term in (2.18)] behaves like λ^2/ω_0 and the quantities (2.21) and (2.22) both become unity.

In conclusion, only the exponential law survives in limit (1.1), with the correct normalization factor ($\mathcal{Z} = 1$), and one is able to derive a purely exponential behavior

(Markovian dynamics) from the quantum mechanical Schrödinger equation (unitary dynamics). It is important to notice that, in order to obtain the exponential law, a normalizable state (such as a wave packet) must be taken as initial state. Our initial state $|2; 0\rangle$ is indeed normalizable: see (2.8).

3.1. Two-level atom in the rotating-wave approximation

Let us now proceed to a more formal analysis. Write the evolution operator as

$$U(t) = \frac{i}{2\pi} \int_C dE \frac{e^{-iEt}}{E - H}, \quad (3.7)$$

where the path C is a straight horizontal line just above the real axis (this is the equivalent of the Bromwich path in the Laplace plane). By defining the resolvents ($\Im E > 0$)

$$S(E) \equiv \left\langle 2; 0 \left| \frac{1}{E - H_0} \right| 2; 0 \right\rangle = \frac{1}{E - \omega_0}, \quad S'(E) \equiv \left\langle 2; 0 \left| \frac{1}{E - H} \right| 2; 0 \right\rangle. \quad (3.8)$$

Dyson's resummation reads

$$S'(E) = S(E) + \lambda^2 S(E) \Sigma^{(2)}(E) S(E) + \lambda^4 S(E) \Sigma^{(2)}(E) S(E) \Sigma^{(2)}(E) S(E) + \dots, \quad (3.9)$$

where $\Sigma^{(2)}(E) = \langle 2; 0 | V (E - H_0)^{-1} V | 2; 0 \rangle$ is the 1-particle irreducible self-energy function. In the rotating-wave approximation $\Sigma^{(2)}(E)$ consists only of a second-order diagram and can be evaluated exactly:

$$\Sigma^{(2)}(E) \equiv \int_0^\infty d\omega \frac{|\varphi(\omega)|^2}{E - \omega} = i\lambda Q(-iE/\lambda), \quad (3.10)$$

where the matrix element $\varphi = \varphi_{\beta}$ in (2.6) and Q is the function in (2.16). In the complex E -plane $\Sigma^{(2)}(E)$ has a branch cut running from 0 to ∞ , a branching point in the origin and no singularity on the first Riemann sheet. Summing the series

$$S'(E) = \frac{1}{S(E)^{-1} - \lambda^2 \Sigma^{(2)}(E)} = \frac{1}{E - \omega_0 - \lambda^2 \Sigma^{(2)}(E)}, \quad (3.11)$$

we obtain for the survival amplitude

$$\begin{aligned} \mathcal{A}(t) &\equiv \langle 2; 0 | e^{iH_0 t} U(t) | 2; 0 \rangle = \frac{i}{2\pi} \int_C dE e^{-iEt} S'(E + \omega_0) \\ &= \frac{i}{2\pi} \int_C dE \frac{e^{-iEt}}{E - \lambda^2 \Sigma^{(2)}(E + \omega_0)}. \end{aligned} \quad (3.12)$$

In Van Hove's limit one looks at the evolution of the system over time intervals of order $t = \tilde{t}/\lambda^2$ (\tilde{t} independent of λ), in the limit of small λ . Our purpose is to see how

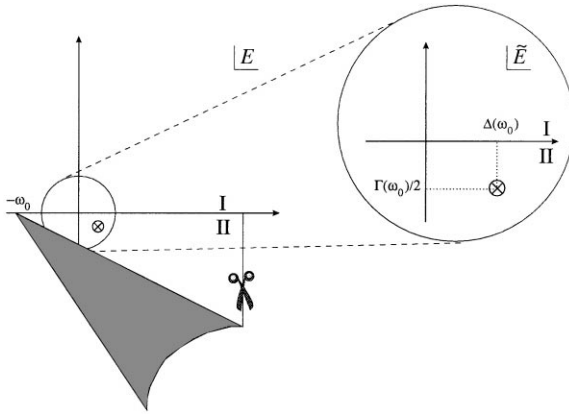


Fig. 1. Singularities of propagator (3.12) in the complex- E plane. The first Riemann sheet (I) is singularity free. The logarithmic cut is due to $\Sigma^{(2)}(E)$ and the pole is located on the second Riemann sheet (II). In the complex- \tilde{E} plane, the pole has coordinates (3.17)–(3.18).

this limit works in the complex-energy plane, i.e. what is the limiting form of the propagator. To this end, by rescaling time $\tilde{t} \equiv \lambda^2 t$, we can write

$$\mathcal{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \frac{i}{2\pi} \int_C d\tilde{E} \frac{e^{-i\tilde{E}\tilde{t}}}{\tilde{E} - \Sigma^{(2)}(\lambda^2\tilde{E} + \omega_0)}, \tag{3.13}$$

where we are naturally led to introduce the rescaled energy $\tilde{E} \equiv E/\lambda^2$. Taking the Van Hove limit we get

$$\tilde{\mathcal{A}}(\tilde{t}) \equiv \lim_{\lambda \rightarrow 0} \mathcal{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \frac{i}{2\pi} \int_C d\tilde{E} e^{-i\tilde{E}\tilde{t}} \tilde{S}'(\tilde{E}), \tag{3.14}$$

where the propagator in the rescaled energy reads

$$\tilde{S}'(\tilde{E}) \equiv \lim_{\lambda \rightarrow 0} \frac{1}{\tilde{E} - \Sigma^{(2)}(\lambda^2\tilde{E} + \omega_0)} = \frac{1}{\tilde{E} - \Sigma^{(2)}(\omega_0 + i0^+)}, \tag{3.15}$$

the term $+i0^+$ being due to the fact that $\Im \tilde{E} > 0$. The self-energy function in the $\lambda \rightarrow 0$ limit becomes

$$\Sigma^{(2)}(\omega_0 + i0^+) = - \int_0^\infty d\omega \frac{|\varphi(\omega)|^2}{\omega - \omega_0 - i0^+} = A(\omega_0) - \frac{i}{2}\Gamma(\omega_0), \tag{3.16}$$

where

$$A(\omega_0) \equiv \mathcal{P} \int_0^\infty d\omega \frac{|\varphi(\omega)|^2}{\omega_0 - \omega}, \tag{3.17}$$

$$\Gamma(\omega_0) \equiv 2\pi|\varphi(\omega_0)|^2 \tag{3.18}$$

which yields a purely exponential decay (Weisskopf–Wigner approximation and Fermi Golden Rule). In Fig. 1 we endeavoured to clarify the role played by the time–energy rescaling in the complex- E plane.

One can get a more detailed understanding of the mechanisms that underpin the limiting procedure by looking at higher-order terms in the coupling constant. The pole of the original propagator (3.11) satisfies the equation

$$E_{\text{pole}} - \lambda^2 \Sigma^{(2)}(E_{\text{pole}} + \omega_0) = 0 \tag{3.19}$$

which can be solved by expanding the self-energy function around $E = \omega_0$ in power series

$$\Sigma^{(2)}(E + \omega_0) = \Sigma^{(2)}(\omega_0) + E \Sigma^{(2)'}(\omega_0) + \frac{E^2}{2} \Sigma^{(2)''}(\omega_0) + \dots \tag{3.20}$$

whose radius of convergence is ω_0 , due to the branching point of $\Sigma^{(2)}$ in the origin. We get (iteratively)

$$E_{\text{pole}} = \lambda^2 \Sigma^{(2)}(\omega_0) + \lambda^4 \Sigma^{(2)'}(\omega_0) \Sigma^{(2)}(\omega_0) + O(\lambda^6) \tag{3.21}$$

which, due to (3.16), becomes

$$E_{\text{pole}} \equiv \Delta E - \frac{i}{2} \gamma = \lambda^2 \Delta(\omega_0) - i \frac{\lambda^2}{2} \Gamma(\omega_0) + O(\lambda^4). \tag{3.22}$$

In the rescaled energy (3.22) reads

$$\tilde{E}_{\text{pole}} = \frac{E_{\text{pole}}}{\lambda^2} = \Delta(\omega_0) - \frac{i}{2} \Gamma(\omega_0) + O(\lambda^2) \xrightarrow{\lambda \rightarrow 0} \Delta(\omega_0) - \frac{i}{2} \Gamma(\omega_0) \tag{3.23}$$

which is the same as (3.16). This is again the Fermi Golden Rule.

3.2. *N-level atom with counter-rotating terms*

Before proceeding to a general analysis it is interesting to see how the above model is modified by the presence of the other atomic levels and the inclusion of counter-rotating terms in the interaction Hamiltonian. This will enable us to pin down other salient features of the $\lambda^2 t$ limit. The Hamiltonian is

$$H = H'_0 + \lambda V', \tag{3.24}$$

where

$$H'_0 \equiv \sum_v \omega_v b_v^\dagger b_v + \sum_\beta \int_0^\infty d\omega \omega a_{\omega\beta}^\dagger a_{\omega\beta}, \tag{3.25}$$

$$V' = \sum_{\mu, \nu} \sum_\beta \int_0^\infty d\omega [\varphi_\beta^{\mu\nu}(\omega) b_\mu^\dagger b_\nu a_{\omega\beta}^\dagger + \varphi_\beta^{\mu\nu*}(\omega) b_\nu^\dagger b_\mu a_{\omega\beta}], \tag{3.26}$$

where ν runs over all the atomic states and b_ν^\dagger, b_ν and $a_{\omega\beta}^\dagger, a_{\omega\beta}$ satisfy anticommutation and commutation relations, respectively. [Hamiltonian (2.1)–(2.3) is recovered if we set $\omega_2 = \omega_0$, $\omega_1 = 0$ and neglect the counter-rotating terms.] Starting from the initial state $|\mu; 0\rangle$, Dyson’s resummation yields

$$S'(E) = \frac{1}{S(E)^{-1} - \lambda^2 \Sigma(E)} = \frac{1}{E - \omega_\mu - \lambda^2 \Sigma(E)} \tag{3.27}$$

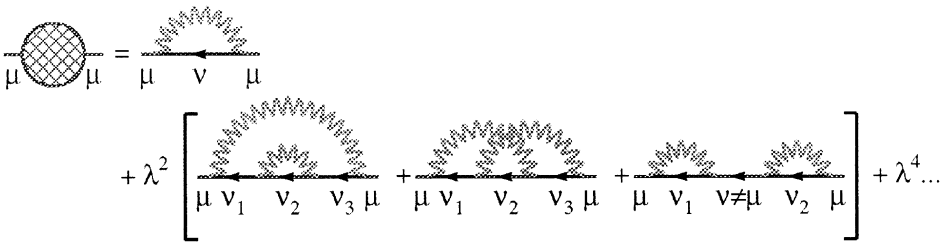


Fig. 2. Graphic representation of (3.28): $\Sigma^{(2)}$ and $\Sigma^{(4)}$ are in the first and second line, respectively.

and the 1-particle irreducible self-energy function takes the form

$$\Sigma(E) = \Sigma^{(2)}(E) + \lambda^2 \Sigma^{(4)}(E) + \dots \tag{3.28}$$

with

$$\Sigma^{(2)}(E) \equiv \sum_{\nu, \beta} \int_0^\infty d\omega \frac{|\varphi_\beta^{\nu\mu}(\omega)|^2}{E - \omega_\nu - \omega}. \tag{3.29}$$

Both $\Sigma^{(2)}$ and $\Sigma^{(4)}$ are shown as Feynman diagrams in Fig. 2. In the Van Hove limit one obtains

$$\Sigma(\lambda^2 \tilde{E} + \omega_\mu) \xrightarrow{\lambda \rightarrow 0} \Sigma^{(2)}(\lambda^2 \tilde{E} + \omega_\mu)|_{\lambda=0} = \Sigma^{(2)}(\omega_\mu + i0^+). \tag{3.30}$$

The propagator in the rescaled energy now takes the form

$$\tilde{S}'(\tilde{E}) = \lim_{\lambda \rightarrow 0} \frac{1}{\tilde{E} - \Sigma^{(2)}(\lambda^2 \tilde{E} + \omega_\mu) + O(\lambda^2)} = \frac{1}{\tilde{E} - \Sigma^{(2)}(\omega_\mu + i0^+)}, \tag{3.31}$$

where

$$\Sigma^{(2)}(\omega_\mu + i0^+) = \sum_{\nu, \beta} \int_0^\infty d\omega \frac{|\varphi_\beta^{\nu\mu}(\omega)|^2}{\omega_\mu - \omega_\nu - \omega + i0^+}. \tag{3.32}$$

The last two equations correspond to (3.15)–(3.16): the propagator reduces to that of a generalized rotating-wave approximation.

We see that the Van Hove limit works by following two logical steps. First, it constrains the evolution in a Tamm–Dancoff sector: the system can only “explore” those states that are directly related to the initial state μ by the interaction V' . In other words, in this limit, the “excitation number” $\mathcal{N}_\mu \equiv b_\mu^\dagger b_\mu + \sum_{\beta, \omega} a_{\omega\beta}^\dagger a_{\omega\beta}$ becomes a conserved quantity (even though the original Hamiltonian contains counter-rotating terms) and, as a consequence, the self-energy function consists only of a second-order contribution that can be evaluated exactly. Second, it reduces this second-order contribution, which depends on energy as in (3.29), to a constant (its value in the energy ω_μ of the initial state), like in (3.30). Hence the analytical properties of the propagator, which had branch-cut singularities, reduce to those of a single complex pole, whose imaginary part (responsible for exponential decay) yields the Fermi Golden rule, evaluated at second order of perturbation theory.

Notice that it is the latter step (and not the former one) which is strictly necessary to obtain a dissipative behavior: Indeed, substitution of the pole value in the total self-energy function yields exponential decay, including, as is well known, higher-order corrections to the Fermi Golden Rule. On the other hand, the first step is very important when one is interested in computing the leading-order corrections to the exponential behavior. To this purpose one can solve the problem in a restricted Tamm–Duncoff sector of the total Hilbert space (i.e., in an eigenspace of \mathcal{N}_μ – in our case, $\mathcal{N}_\mu = 1$) and exactly evaluate the evolution of the system with its deviations from exponential law.

Let us add a final remark. As is well known, a nondispersive propagator yields a Markovian evolution. Let us briefly sketch how this occurs in the present model. From (3.27), antitransforming,

$$\frac{i}{2\pi} \int_C dE e^{-iEt} (ES'(E + \omega_\mu) - 1) = \frac{i}{2\pi} \int_C dE e^{-iEt} \lambda^2 \Sigma(E + \omega_\mu) S'(E + \omega_\mu), \quad (3.33)$$

we obtain (for $t > 0$)

$$i\dot{\mathcal{A}}(t) = \lambda^2 \int_0^t d\tau \sigma(t - \tau) \mathcal{A}(\tau), \quad (3.34)$$

where $\mathcal{A}(t)$ is the survival amplitude (2.10) and

$$\sigma(t) \equiv \frac{1}{2\pi} \int_C dE e^{-iEt} \Sigma(E + \omega_\mu) = \frac{e^{i\omega_\mu t}}{2\pi} \int_C dE e^{-iEt} \Sigma(E). \quad (3.35)$$

Eq. (3.34) is clearly nonlocal in time and all memory effects are contained in $\sigma(t)$, which is the antitransform of the self-energy function. If such a self-energy function is a complex constant (energy independent), $\Sigma(E) = C$, then $\sigma(t) = C\delta(t)$ and Eq. (3.34) becomes

$$i\dot{\mathcal{A}}(t) = \lambda^2 C \mathcal{A}(t), \quad (3.36)$$

describing a Markovian behavior, without memory effects [29,30]. In particular, the Van Hove limit is equivalent to set $C = \Sigma^{(2)}(\omega_\mu + i0^+)$ and the Weisskopf–Wigner approximation is $C = \Sigma^{(2)}(\omega_\mu + i0^+) + O(\lambda^2)$.

In conclusion, in the Van Hove limit, the evolution of our system, which was non-local in time due to the dispersive character of the propagator (the self-energy function depended on E) becomes local and Markovian (only the value of the self-energy function in ω_μ determines the evolution).

4. General framework

We can now further generalize our analysis: consider the Hamiltonian

$$H = H_0 + \lambda V \quad (4.1)$$

and suppose that the initial state $|a\rangle$ has the following properties:

$$\begin{aligned} H_0|a\rangle &= E_a|a\rangle, & \langle a|V|a\rangle &= 0, \\ \langle a|a\rangle &= 1. \end{aligned} \tag{4.2}$$

The survival amplitude of state $|a\rangle$ reads

$$\begin{aligned} \mathcal{A}(t) &\equiv \langle a|e^{iH_0t}U(t)|a\rangle = \frac{i}{2\pi} \int_C dE e^{-iEt} S'(E + E_a) \\ &= \frac{i}{2\pi} \int_C dE \frac{e^{-iEt}}{E - \lambda^2 \Sigma(E + E_a)}, \end{aligned} \tag{4.3}$$

where $S'(E) \equiv \langle a|(E - H)^{-1}|a\rangle$ and $\Sigma(E)$ is the 1-particle irreducible self-energy function, that can be expressed by a perturbation expansion

$$\lambda^2 \Sigma(E) = \lambda^2 \Sigma^{(2)}(E) + \lambda^4 \Sigma^{(4)}(E) + \dots \tag{4.4}$$

The second-order contribution has the general form

$$\begin{aligned} \Sigma^{(2)}(E) &\equiv \left\langle a \left| VP_d \frac{1}{E - H_0} P_d V \right| a \right\rangle = \sum_{n \neq a} |\langle a|V|n\rangle|^2 \frac{1}{E - E_n} \\ &= \int_0^\infty \frac{dE'}{2\pi} \frac{\Gamma(E')}{E - E'}, \end{aligned} \tag{4.5}$$

where $P_d = 1 - |a\rangle\langle a|$ is the projector over the decayed states, $\{|n\rangle\}$ is a complete set of eigenstates of H_0 ($H_0|n\rangle = E_n|n\rangle$) and we set $E_0 = 0$) and

$$\Gamma(E) \equiv 2\pi \sum_{n \neq a} |\langle a|V|n\rangle|^2 \delta(E - E_n). \tag{4.6}$$

Notice that $\Gamma(E) \geq 0$ for $E > 0$ and is zero otherwise. In the Van Hove limit we get

$$\tilde{\mathcal{A}}(\tilde{t}) \equiv \lim_{\lambda \rightarrow 0} \mathcal{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \frac{i}{2\pi} \int_C d\tilde{E} e^{-i\tilde{E}\tilde{t}} \tilde{S}'(\tilde{E}), \tag{4.7}$$

where the resulting propagator in the rescaled energy $\tilde{E} = E/\lambda^2$ reads

$$\tilde{S}'(\tilde{E}) = \frac{1}{\tilde{E} - \Sigma^{(2)}(E_a + i0^+)}. \tag{4.8}$$

To obtain this result we used

$$\Sigma(\lambda^2 \tilde{E} + E_a) \xrightarrow{\lambda \rightarrow 0} \Sigma^{(2)}(\lambda^2 \tilde{E} + E_a)|_{\lambda=0} = \Sigma^{(2)}(E_a + i0^+) \tag{4.9}$$

(Weisskopf–Wigner approximation and Fermi Golden Rule).

Just above the positive real axis we can write

$$\Sigma^{(2)}(E + i0^+) = \Lambda(E) - \frac{i}{2} \Gamma(E), \tag{4.10}$$

where

$$\Lambda(E) = \mathcal{P} \int_0^\infty \frac{dE'}{2\pi} \frac{\Gamma(E')}{E - E'}. \tag{4.11}$$

Let $\Gamma(E)$ be summable in $(0, +\infty)$. Then

$$\Gamma(E) \propto E^{\eta-1} \quad \text{for } E \rightarrow 0 \quad (4.12)$$

for some $\eta > 0$, and one gets the following asymptotic behavior at short and long times:

$$P(t) \sim 1 - \frac{t^2}{\tau_Z^2} \quad (t \ll \tau_Z), \quad (4.13)$$

$$P(t) \sim |Z|^2 e^{-t/\tau_E} + \lambda^4 \frac{|C|^2}{(E_a t)^{2\eta}} + 2\lambda^2 \frac{|CZ|}{(E_a t)^\eta} e^{-t/2\tau_E} \cos[(E_a + \Delta E)t - \zeta] \quad (t \gg \tau_Z), \quad (4.14)$$

where

$$\tau_Z = \frac{1}{\lambda} \left[\int_0^\infty \frac{dE}{2\pi} \Gamma(E) \right]^{-1/2}, \quad (4.15)$$

$$\tau_E = \frac{1}{\lambda^2 \Gamma(E_a)}, \quad (4.16)$$

$$\Delta E = \lambda^2 \Delta(E_a), \quad (4.17)$$

$$\zeta = \text{Arg } Z - \text{Arg } C, \quad (4.18)$$

$$Z = 1 + O(\lambda^2), \quad (4.19)$$

$$C = 1 + O(\lambda^2). \quad (4.20)$$

The transition to a power law occurs when the first two terms in the r.h.s. of (4.14) are comparable, namely for $t = \tau_{\text{pow}}$, where τ_{pow} is solution of the equation

$$\frac{\tau_{\text{pow}}}{\tau_E} = 4(\eta + 1) \log \frac{1}{\lambda} + 2\eta \log \frac{E_a}{\Gamma(E_a)} + \log \left| \frac{Z}{C} \right| + \eta \log \frac{\tau_{\text{pow}}}{\tau_E}, \quad (4.21)$$

i.e., for $\lambda \rightarrow 0$

$$\tau_{\text{pow}} = 4\tau_E(\eta + 1) \log \lambda^{-1} + O(\log \log \lambda^{-1}). \quad (4.22)$$

Let us now look at the temporal behavior for a small but *finite* value of λ , using Van Hove's technique. In the rescaled time, $\tilde{t} = \lambda^2 t$, the Zeno region vanishes

$$\tilde{\tau}_Z \equiv \lambda^2 \tau_Z = \lambda \left[\int_0^\infty \frac{dE}{2\pi} \Gamma(E) \right]^{-1/2} = O(\lambda) \quad (4.23)$$

and Eq. (4.14) becomes valid at *shorter and shorter* (rescaled) times and reads

$$P(\tilde{t}) \sim |Z|^2 e^{-\tilde{t}/\tilde{\tau}_E} + \lambda^{4(\eta+1)} \frac{|C|^2}{(E_a \tilde{t})^{2\eta}} + 2\lambda^{2(\eta+1)} \frac{|CZ|}{(E_a \tilde{t})^\eta} e^{-\tilde{t}/2\tilde{\tau}_E} \cos \left(\frac{E_a + \Delta E}{\lambda^2} \tilde{t} - \zeta \right), \quad (4.24)$$

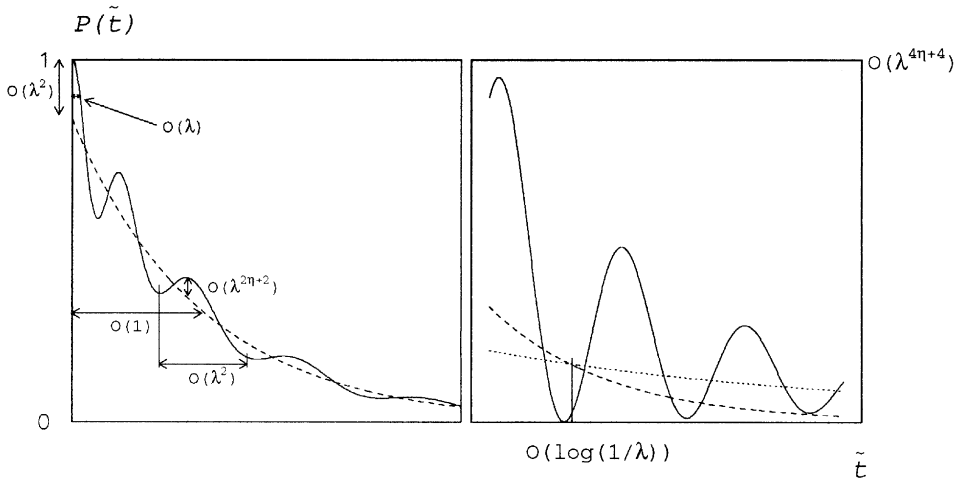


Fig. 3. Essential features (not in scale!) of the survival probability as a function of the rescaled time \tilde{t} . The Zeno time is $O(\lambda)$, the lifetime $O(1)$, during the whole evolution there are oscillations of amplitude $O(\lambda^{2\eta+2})$ and the transition to a power law occurs after a time $O(\log(1/\lambda))$ [see (4.23)–(4.26)]. From (4.19), the normalization factor becomes unity like $1 - O(\lambda^2)$. The dashed line is the exponential and the dotted line the power law.

where

$$\tilde{\tau}_E \equiv \lambda^2 \tau_E = \frac{1}{\Gamma(E_a)} = O(1), \tag{4.25}$$

$$\tilde{\tau}_{\text{pow}} \equiv \lambda^2 \tau_{\text{pow}} \simeq 4\tilde{\tau}_E(\eta + 1) \log \frac{1}{\lambda} = O\left(\log \frac{1}{\lambda}\right). \tag{4.26}$$

Fig. 3 displays the main features of the temporal behavior of the survival probability. The typical values of the physical constants [see for instance (2.9)] yield very small deviations from the exponential law. For this reason, we displayed in Fig. 3 the survival probability by greatly exaggerating its most salient features.

The Van Hove limit performs several actions at once: It makes the initial quadratic (quantum Zeno) region vanish, it “squeezes” out the oscillations and it “pushes” the power law to infinity, leaving only a clean exponential law at all times, with the right normalization. All this is not surprising, being implied by the Weisskopf–Wigner approximation. However, the concomitance of these features is so remarkable that one cannot but wonder at the effectiveness of this limiting procedure. In atomic and molecular physical systems the smallness of the coupling constant and other physical parameters makes the experimental observation of deviations from exponential a very difficult task (see for example the simple model investigated in Section 2). The eventuality that alternative physical systems might exhibit experimentally observable non-exponential decays, as well as the possibility of modifying the lifetimes of unstable systems by means of intense laser beams [46,47] are at present under investigation.

References

- [1] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Oxford Univ. Press, London, 1958.
- [2] A. Beskow, J. Nilsson, *Ark. Fys.* 34 (1967) 561.
- [3] L.A. Khalifin, *Zh. Eksp. Teor. Fiz. Pisma. Red.* 8 (1968) 106 [*JETP Lett.* 8 (1968) 65].
- [4] L. Fonda, G.C. Ghirardi, A. Rimini, T. Weber, *Nuovo Cimento A* 15 (1973) 689.
- [5] L. Fonda, G.C. Ghirardi, A. Rimini, T. Weber, *Nuovo Cimento A* 18 (1973) 805.
- [6] B. Misra, E.C.G. Sudarshan, *J. Math. Phys.* 18 (1977) 756.
- [7] A. Peres, *Am. J. Phys.* 48 (1980) 931.
- [8] A. Peres, *Ann. Phys.* 129 (1980) 33.
- [9] L. Mandelstam, I. Tamm, *J. Phys.* 9 (1945) 249.
- [10] V. Fock, N. Krylov, *J. Phys.* 11 (1947) 112.
- [11] E.J. Hellund, *Phys. Rev.* 89 (1953) 919.
- [12] M. Namiki, N. Mugibayashi, *Prog. Theor. Phys.* 10 (1953) 474.
- [13] L.A. Khalifin, *Dokl. Acad. Nauk USSR* 115 (1957) 277 [*Sov. Phys. Dokl.* 2 (1957) 340].
- [14] L.A. Khalifin, *Zh. Eksp. Teor. Fiz.* 33 (1958) 1371 [*Sov. Phys. JETP* 6 (1958) 1053].
- [15] J.J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley, Reading, MA, 1994.
- [16] A. Messiah, *Quantum Mechanics*, Interscience, New York, 1961.
- [17] L.S. Brown, *Quantum Field Theory*, Cambridge University Press, Bristol, 1992.
- [18] H. Nakazato, M. Namiki, S. Pascazio, *Int. J. Mod. Phys. B* 10 (1996) 247.
- [19] E. Fermi, *Rev. Mod. Phys.* 4 (1932) 87.
- [20] E. Fermi, *Nuclear Physics*, Univ. Chicago, Chicago, 1950, pp. 136, 148.
- [21] E. Segré (Ed.), *Notes on Quantum Mechanics. A Course Given at the University of Chicago in 1954*, Univ. Chicago, Chicago, 1960, Lec. 23.
- [22] E.B. Norman, S.B. Gazes, S.G. Crane, D.A. Bennett, *Phys. Rev. Lett.* 60 (1988) 2246.
- [23] G.-C. Cho, H. Kasari, Y. Yamaguchi, *Prog. Theor. Phys.* 90 (1993) 803.
- [24] S.R. Wilkinson et al., *Nature* 387 (1997) 557.
- [25] G. Gamow, *Z. Phys.* 51 (1928) 204.
- [26] V. Weisskopf, E.P. Wigner, *Z. Phys.* 63 (1930) 54.
- [27] V. Weisskopf, E.P. Wigner, *Z. Phys.* 65 (1930) 18.
- [28] G. Breit, E.P. Wigner, *Phys. Rev.* 49 (1936) 519.
- [29] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam, 1992.
- [30] C.W. Gardiner, *Handbook of Stochastic Methods*, Springer, Berlin, 1990.
- [31] L. Van Hove, *Physica* 21 (1955) 517.
- [32] L. Accardi, S.V. Kozyrev, I.V. Volovich, *Phys. Rev. A* 56 (1997) 2557.
- [33] L. Accardi, Y.G. Lu, I.V. Volovich, *Quantum Theory and Its Stochastic Limit*, Oxford University Press, London, in press.
- [34] W. Heitler, *Proc. Cambridge Philos. Soc.* 32 (1936) 112.
- [35] A.I. Akhiezer, V.B. Berestetskii, *Quantum Electrodynamics*, Interscience Publ., New York, 1965.
- [36] H.E. Moses, *Lett. Nuovo Cimento* 4 (1972) 51.
- [37] H.E. Moses, *Phys. Rev. A* 8 (1973) 1710.
- [38] J. Seke, *Physica A* 203 (1994) 269, 284.
- [39] P. Facchi, S. Pascazio, *Phys. Lett. A* 241 (1998) 139.
- [40] P.L. Knight, P.W. Milonni, *Phys. Lett.* 56 A (1976) 275.
- [41] L. Davidovich, H.M. Nussenzveig, in: A.O. Barut (Ed.), *Foundations of Radiation Theory and Quantum Electrodynamics*, Plenum, New York, 1980, p. 83.
- [42] M. Hillery, *Phys. Rev. A* 24 (1981) 933.
- [43] J. Seke, W. Herfort, *Phys. Rev. A* 40 (1989) 1926.
- [44] J. Seke, *Phys. Rev. A* 45 (1992) 542.
- [45] N.A. Enaki, *Zh. Eksp. Teor. Fiz.* 109 (1996) 1130 [*Sov. Phys. JETP* 82 (1996) 607].
- [46] E. Mihokova, S. Pascazio, L.S. Schulman, *Phys. Rev. A* 56 (1997) 25.
- [47] P. Facchi, S. Pascazio, *Acta Physica Slovaca* 49 (1999) 557.