

Typical observables in a two-mode Bose system

Paolo Facchi,^{1,2} Saverio Pascazio,^{1,2} Francesco V. Pepe,^{1,2} and Golam Ali Sekh²

¹*Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy*

²*INFN, Sezione di Bari, I-70126 Bari, Italy*

(Received 15 January 2015; published 31 March 2015)

A class of k -particle observables in a two-mode system of Bose particles is characterized by typicality: if the state of the system is sampled out of a suitable ensemble, an experimental measurement of that observable (almost) always yields the same result. We investigate the general features of typical observables and the criteria to determine typicality and finally focus on the case of density correlation functions, which are related to spatial distribution of particles and interference.

DOI: [10.1103/PhysRevA.91.033637](https://doi.org/10.1103/PhysRevA.91.033637)

PACS number(s): 03.75.Dg, 03.75.Hh, 05.30.Jp

I. INTRODUCTION

The experimental realization of Bose-Einstein condensates raised great theoretical interest. A system of Bose particles in one or a few single-particle states (modes) is an important workbench for fundamental concepts in quantum mechanics and statistical physics [1–7]. A large number of particles distributed among two different modes, for example, enables one to perform a full quantum double-slit experiment in a single experimental run [8]. Moreover, it has been proposed that fluctuations in the interference patterns can probe interesting characteristics of many-body systems [9–19].

Interference is an interesting example of a property that *weakly* depends on the choice of the state of the system. In a two-mode system, second-order-interference properties are similar as far as one considers a number state or a phase state [5,6,20], while first-order properties are very different. These features explain why an interference pattern can be experimentally observed in single experimental runs by measuring the particles' positions [1,3,4,20–23], although in the case of number states the offset of the pattern fluctuates randomly, so that averaging over a few experimental runs yields a flat density profile. It is also interesting to note that, when the number of particles is large, there are interference-related observables whose experimental measurement yields the same result at each run with overwhelming probability [24–27].

These considerations lead us to a general definition of *typicality* of an observable. Typicality is a mathematical concept related to the phenomenon of measure concentration [28]. It has been used in many emerging phenomena in physics and other sciences, with interesting applications on the structure of entanglement in large quantum systems [29–33], and the search for a quantum-mechanical justification of some primary statistical-mechanical concepts [34–44]. In this article, we shall apply the notion of typicality to a two-mode Bose system with a fixed number of particles. We will study the properties of observables with respect to uniform sampling of a suitable Hilbert subspace. An observable will be called typical if each experimental run, performed on *any* state of the subspace, would yield the same result with overwhelming probability. According to the probabilistic interpretation of quantum mechanics, the expectation value of an observable provides information on the average of experimental runs on the (pure or mixed) state. We shall build on the results of Ref. [26] and prove that if the observable is typical, there exists

an expectation value that contains information on (almost) every single experimental run, rather than on the average result. Throughout this article, we shall say that a property is “almost” always valid if it is always valid, apart from a negligible set of cases. A number of concrete examples will be discussed in the following.

This article is organized as follows. In Sec. II we specify the properties of the statistical ensemble and introduce the general formalism, which leads to the definition of typicality of an observable. In Sec. III an analysis of the structure of fluctuations for a general k -particle observable is performed. Quantitative criteria will be given to determine typicality and obtain information on higher-order fluctuations. Section IV includes an application of the typicality criteria to density correlation operators, with the case study of counterpropagating plane-wave modes and an extension to expanding modes in the far-field regime.

II. TYPICALITY OF AN OBSERVABLE

Let us consider a system of bosons, without internal degrees of freedom. The system can be described in a second-quantization picture by introducing the annihilation and creation field operators $\hat{\Psi}(\mathbf{r})$ and $\hat{\Psi}^\dagger(\mathbf{r})$, satisfying canonical equal-time commutation relations:

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')] = 0, \quad (1)$$

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}'). \quad (2)$$

Once an orthonormal basis for the single-particle Hilbert space has been chosen, the field operators can be expanded as sums of mode operators, which create or annihilate a particle in one of the basis states. We are going to analyze a system made up of N bosons, distributed among two orthogonal modes, a and b , with single-particle wave functions $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$. A useful basis for the description of the N -particle Hilbert space is formed by the Fock states

$$|\ell\rangle := \left| \left(\frac{N}{2} + \ell \right)_a, \left(\frac{N}{2} - \ell \right)_b \right\rangle, \quad (3)$$

in which the two modes have well-defined occupation numbers. We are assuming that N is even for simplicity, but this specification is immaterial in the large- N limit. In a second-quantized formalism, Fock states are obtained by applying a sequence of mode creation operators to the

vacuum $|\Omega\rangle$:

$$|\ell\rangle = \frac{1}{\sqrt{(N/2 + \ell)!(N/2 - \ell)!}} (\hat{a}^\dagger)^{N/2 + \ell} (\hat{b}^\dagger)^{N/2 - \ell} |\Omega\rangle. \quad (4)$$

Since $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$ are orthonormal, the mode operators

$$\hat{a} = \int d\mathbf{r} \psi_a^*(\mathbf{r}) \hat{\Psi}(\mathbf{r}), \quad \hat{b} = \int d\mathbf{r} \psi_b^*(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \quad (5)$$

satisfy the canonical commutation relations

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \quad (6)$$

and all the operators of mode a commute with those of mode b . The number operators $\hat{N}_a = \hat{a}^\dagger \hat{a}$ and $\hat{N}_b = \hat{b}^\dagger \hat{b}$ count the numbers of particles in each mode.

We are interested in the properties of an ensemble of states randomly sampled from the n -dimensional subspace

$$\mathcal{H}_n = \text{span}\{|\ell\rangle, |\ell| < n/2\}, \quad (7)$$

spanned by the Fock states with $0 < n \leq N + 1$ (n is odd for simplicity.) In the case $n = 1$, the subspace $\mathcal{H}_1 = \text{span}\{|\ell = 0\rangle\}$ is one-dimensional. Thus, only the state $|\ell = 0\rangle = |(N/2)_a, (N/2)_b\rangle$ has nonvanishing probability [12,20–24,45]. In this situation, it is improper to talk of typicality. In the following, we are instead interested in the large- n case [26], where the notion of typicality acquires meaning. If a Bose-Einstein condensate (BEC) made up on N particles is “evenly” split between the two modes (e.g., by adiabatic splitting), one expects $n = O(\sqrt{N})$. However, we shall not restrict our attention to this special case and assume throughout this article the far more general condition

$$n = o(N), \quad (8)$$

namely, $\lim_{N \rightarrow \infty} n/N = 0$.

We shall assume that states are randomly sampled out of space (7) with a *uniform* distribution. This is clearly a simplifying assumption: the number of states that are involved in the description and their amplitudes depends on the experimental procedure, yielding the separation of the condensate in the two modes [46]. However, it will emerge that our main results are qualitatively unchanged for a large and relevant class of probability distributions on \mathcal{H}_n .

The average of the projection on a random vector state $|\Phi_N\rangle \in \mathcal{H}_n$ uniformly sampled on \mathcal{H}_n yields the (microcanonical) density matrix

$$\hat{\rho}_n = \overline{|\Phi_N\rangle\langle\Phi_N|} = \frac{1}{n} \sum_{|\ell| < n/2} |\ell\rangle\langle\ell| =: \frac{1}{n} \hat{P}_n, \quad (9)$$

where \hat{P}_n is the projection onto the subspace \mathcal{H}_n . In the following, only the density matrix (9) will explicitly appear in our analysis and calculation. The definition and use of the random state $|\Phi_N\rangle$ in Eq. (9) is superfluous and can be dispensed with, in accord with the prescription of Ockham’s razor. In this respect, it is worth stressing that no hypothesis of decoherence will be made, and if one wants, one can safely assume that in each experimental run a wave function describes the condensate, which is therefore in a pure state.

Given an observable \hat{A} , the statistical average of its expectation value reads

$$\bar{A} := \text{Tr}(\hat{\rho}_n \hat{A}) = \frac{1}{n} \sum_{|\ell| < n/2} \langle\ell|\hat{A}|\ell\rangle. \quad (10)$$

Its quantum variance is

$$\delta A^2 := \text{Tr}(\hat{\rho}_n \hat{A}^2) - [\text{Tr}(\hat{\rho}_n \hat{A})]^2. \quad (11)$$

The significance of this quantity in the context of ensemble statistics becomes clear once it is decomposed in the sum of two contributions [26,27,47]: it contains both the classical and quantum uncertainties of the observable A in the microcanonical state $\hat{\rho}_n$. Therefore, the asymptotic condition

$$\delta A = o(\bar{A}) \quad (12)$$

for $N \rightarrow \infty$, that is, $\lim_{N \rightarrow \infty} \delta A / \bar{A} = 0$, ensures that in the overwhelming majority of cases the experimental measurement of the observable \hat{A} will fluctuate within an extremely narrow range around the average expectation value \bar{A} . Thus, the outcome of a measurement of \hat{A} is almost always the same (and it equals its average) for every experimental run in the ensemble. We call this property *typicality of the observable*.

In the following sections we will define and characterize conditions for a k -particle observable to be typical and analyze in detail the case of spatial correlation functions, which are related to interference.

III. CONTROL OF FLUCTUATIONS FOR A k -PARTICLE OPERATOR

In a second quantization formalism, the field operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ can be used to build up many-body observables [48]. The simplest ones are the Hermitian $2k$ -point functions

$$\hat{G}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) := \prod_{i=1}^k \hat{\Psi}^\dagger(\mathbf{r}_i) \prod_{j=1}^k \hat{\Psi}(\mathbf{r}_j). \quad (13)$$

Following Eq. (10), one can observe that the ensemble average of \hat{G}_k can be computed for any n once its expectation value over the N -particle two-mode Fock states $|\ell\rangle$ is known. Thus, when the field operators in (13) are expanded in orthogonal modes, only the terms formed by operators associated with modes a and b are relevant in $\overline{\hat{G}_k}$. Moreover, operator products with a different number of \hat{a}^\dagger ’s and \hat{a} ’s (respectively, \hat{b}^\dagger ’s and \hat{b} ’s) give vanishing contributions to $\langle\ell|\hat{G}_k|\ell\rangle$. The relevant terms in (13) can be reordered to be expressed as number operators $\hat{N}_{a,b}$. The average eventually reads [we will assume $k = O(1)$]

$$\begin{aligned} \overline{\hat{G}_k(\mathbf{r}_1, \dots, \mathbf{r}_k)} &= \sum_{m=0}^k F_m(\mathbf{r}_1, \dots, \mathbf{r}_k) \frac{1}{n} \\ &\times \sum_{|\ell| < n/2} \prod_{A=0}^{m-1} \binom{N}{2} + \ell - A \prod_{B=0}^{k-m-1} \binom{N}{2} - \ell - B, \end{aligned} \quad (14)$$

with

$$F_m(\mathbf{r}_1, \dots, \mathbf{r}_k) = |\Phi_m(\mathbf{r}_1, \dots, \mathbf{r}_k)|^2, \quad (15)$$

where $\Phi_m(\mathbf{r}_1, \dots, \mathbf{r}_k)$ is, apart from a normalization factor, the symmetrized k -body wave function with m particles in mode a and $k - m$ particles in mode b :

$$\begin{aligned} \Phi_m(\mathbf{r}_1, \dots, \mathbf{r}_k) &= \sum_{\sigma} \psi_a(\mathbf{r}_{\sigma(1)}) \cdots \psi_a(\mathbf{r}_{\sigma(m)}) \psi_b(\mathbf{r}_{\sigma(m+1)}) \cdots \psi_b(\mathbf{r}_{\sigma(k)}), \end{aligned} \quad (16)$$

with σ denoting permutation of k elements.

An important class of k -particle observables can be obtained by integrating the $2j$ -point functions, for $j \leq k$, with a multiplicative kernel \mathcal{A}_j [49]:

$$\hat{A}_k = \sum_{j=0}^k \int d\mathbf{r}_1 \cdots d\mathbf{r}_j \mathcal{A}_j(\mathbf{r}_1, \dots, \mathbf{r}_j) \hat{G}_j(\mathbf{r}_1, \dots, \mathbf{r}_j). \quad (17)$$

The ensemble average of \hat{A} can be immediately computed by inserting the general results (14). Thus, since $\overline{G_j} = O(N^j)$ when $k = O(1)$, we have

$$\begin{aligned} \overline{A_k} &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_k \mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) \overline{G_k(\mathbf{r}_1, \dots, \mathbf{r}_k)} \\ &\quad + O(N^{k-1}). \end{aligned} \quad (18)$$

In order to study the behavior of the variance (11) and determine whether \hat{A}_k is typical or not, we should compute its square, which involves at the highest order in N a product of \hat{G}_k functions that can be recast into a single $4k$ -point function by normal ordering the field operators:

$$\begin{aligned} \hat{A}_k^2 &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_{2k} [\mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) \mathcal{A}_k(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) \\ &\quad \times \hat{G}_{2k}(\mathbf{r}_1, \dots, \mathbf{r}_{2k})] + O(N^{2k-1}). \end{aligned} \quad (19)$$

From (17)–(19) we get an expression for the variance,

$$\begin{aligned} \delta A_k^2 &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_{2k} [\mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) \mathcal{A}_k(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) \\ &\quad \times \gamma_k(\mathbf{r}_1, \dots, \mathbf{r}_{2k})] + O(N^{2k-1}), \end{aligned} \quad (20)$$

with

$$\begin{aligned} \gamma_k(\mathbf{r}_1, \dots, \mathbf{r}_{2k}) &:= \overline{G_{2k}(\mathbf{r}_1, \dots, \mathbf{r}_{2k})} \\ &\quad - \overline{G_k(\mathbf{r}_1, \dots, \mathbf{r}_k)} \overline{G_k(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k})}, \end{aligned} \quad (21)$$

which depends only on the ensemble averages of G_k and G_{2k} . Notice that the order N^{2k-1} comes both from contributions in (17) with $j < k$ and from normal ordering.

Our aim is to find whether, for some choice of the sampled Hilbert subspace \mathcal{H}_n , the standard deviation (20) is of smaller order with respect to the average $\overline{A_k}$ in the asymptotic $N \rightarrow \infty$ regime. This behavior is in accord with the definition of typicality of the observable \hat{A}_k given in Sec. II. First, let us observe that, since $\overline{G_k}$ is polynomial in N and n , inserting the general result (14) into δA_k^2 yields a polynomial function of degree $2k$ in the number of particles and the dimension of the sampled subspace:

$$\delta A_k^2 = \sum_{p=0}^k \sum_{q=0}^{2k-p} D_{p,q}^{(A_k)} \left(\frac{N}{2}\right)^p n^q. \quad (22)$$

Due to the symmetry of the summand around $\ell = 0$, only terms $N^{2k-q} n^q$ with *even* q are present in (22). It is evident that, as far as $n = o(N)$, it is necessary and sufficient for \hat{A}_k to be typical that

$$\begin{aligned} D_{2k,0}^{(A_k)} &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_{2k} \mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) \mathcal{A}_k(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) \\ &\quad \times \left[\sum_{M=0}^k F_M(\mathbf{r}_1, \dots, \mathbf{r}_{2k}) \right. \\ &\quad \left. - \sum_{m,m'=0}^k F_{m'}(\mathbf{r}_1, \dots, \mathbf{r}_k) F_m(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) \right] \end{aligned} \quad (23)$$

vanishes, namely,

$$D_{2k,0}^{(A_k)} = 0. \quad (24)$$

In this case, since $\overline{A_k} = O(N^k)$, the relative fluctuations are

$$\frac{\delta A_k}{\overline{A_k}} = O\left(\frac{1}{\sqrt{N}}\right) + O\left(\frac{n}{N}\right) \rightarrow 0 \quad \text{for } N \rightarrow \infty, \quad (25)$$

which is consistent with Eq. (12). Fluctuations that scale like $N^{-1/2}$, related to normal ordering and to the very definition of \hat{A}_k , are ensemble independent in the sense that they are present even in degenerate distributions of states. Linear fluctuations in n are clearly related to the dimension of the sampled subspace and therefore strongly depend on the definition of the ensemble. It is interesting to note that, if $n = O(N^\alpha)$, two qualitative regimes can be distinguished: (i) when $\alpha \leq 1/2$, the relative fluctuations scale like $N^{-1/2}$, while (ii) if $\alpha > 1/2$, they asymptotically vanish like $N^{\alpha-1}$, i.e., more slowly. This reasoning is based on the assumption that $D_{2k-2,2}^{(A_k)}$ does not vanish: we will see in the following how this condition can be checked and discuss in the next section a relevant class of exceptions.

Due to the form of the symmetrized products of wave functions (16), the typicality condition (24) can be recast in a more convenient form: in particular, we can dispose of the integral over $2k$ variables by observing that the functions $\Phi_M(\mathbf{r}_1, \dots, \mathbf{r}_{2k})$ can be decomposed as

$$\begin{aligned} \Phi_M(\mathbf{r}_1, \dots, \mathbf{r}_{2k}) &= \sum_{m=\max\{0, M-k\}}^{\min\{M, k\}} \Phi_{M-m}(\mathbf{r}_1, \dots, \mathbf{r}_k) \Phi_m(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}). \end{aligned} \quad (26)$$

This result is related to the cluster decomposition principle in Bose systems [50]. When one computes the square modulus $F_M = |\Phi_M|^2$ of (26), which enters the variance in (21), the term $\sum_m F_{M-m} F_m$ appears. This contribution exactly cancels with the subtracted terms in (21) since it can be obtained by a change of summation indices,

$$\begin{aligned} \sum_{m,m'=0}^k F_{m'}(\mathbf{r}_1, \dots, \mathbf{r}_k) F_m(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) &= \sum_{M=0}^k \sum_{m=0}^M F_{M-m}(\mathbf{r}_1, \dots, \mathbf{r}_k) F_m(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}) \end{aligned}$$

$$+ \sum_{M=k+1}^{2k} \sum_{m=M-k}^k F_{M-m}(\mathbf{r}_1, \dots, \mathbf{r}_k) F_m(\mathbf{r}_{k+1}, \dots, \mathbf{r}_{2k}). \quad (27)$$

Accordingly, a compact form of the typicality condition (24) reads

$$D_{2k,0}^{(A_k)} = \sum_{M=0}^{2k} \sum_{m' \neq m = \max\{0, M-k\}}^{\min\{M, k\}} \mathcal{I}_{M-m, M-m'}^{(A_k)} \mathcal{I}_{m, m'}^{(A_k)} = 0, \quad (28)$$

where the coefficients $\mathcal{I}_{m, m'}$ are integrals over k position variables of products of the type $\Phi_{m'}^* \Phi_m$:

$$\mathcal{I}_{m, m'}^{(A_k)} = \int d\mathbf{r}_1 \cdots d\mathbf{r}_k \mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) (\Phi_{m'}^* \Phi_m)(\mathbf{r}_1, \dots, \mathbf{r}_k). \quad (29)$$

Given the integral kernel \mathcal{A}_k , which determines the highest order in N of the observable \hat{A}_k , and the mode wave functions $\psi_{a,b}(\mathbf{r})$, it is sufficient to compute the integrals (29) to check whether the observable is typical.

It is also interesting to analyze the structure of $D_{2k-2,2}^{(A_k)}$, also in view of the following discussion on spatial correlation function. If $D_{2k-2,2}^{(A_k)}$ is nonvanishing, the transition between an ensemble-independent ($N^{-1/2}$) and an ensemble-dependent behavior of fluctuations occurs for $n = O(N^{1/2})$. In order to analyze this quantity, one should take into account terms of order N^k and $N^{k-2}n^2$ in the general expression (14),

$$\begin{aligned} & \overline{G_k(\mathbf{r}_1, \dots, \mathbf{r}_k)} \\ &= \sum_{m=0}^k F_m(\mathbf{r}_1, \dots, \mathbf{r}_k) \left\{ \left(\frac{N}{2}\right)^k + \left(\frac{N}{2}\right)^{k-2} \right. \\ & \quad \times \left. \frac{n^2}{24} [k^2 - k(4m+1) + 4m^2] \right\} \\ & \quad + O(N^{k-1}) + O(N^{k-4}n^4), \end{aligned} \quad (30)$$

which is to be used in the computation of (21). Integration over the kernel \mathcal{A}_k and application of the same change of indices leading to Eq. (27) yield the result

$$\begin{aligned} D_{2k-2,2}^{(A_k)} &= \frac{1}{12} \left[(\mathcal{J}^{(A_k)})^2 + \sum_{M=0}^{2k} \sum_{m' \neq m = \max\{0, M-k\}}^{\min\{M, k\}} \right. \\ & \quad \times \left. \mathcal{I}_{M-m, M-m'}^{(A_k)} \mathcal{I}_{m, m'}^{(A_k)} \right] \\ & \quad \times [2k^2 - k(4M+1) + 2M^2], \end{aligned} \quad (31)$$

where the \mathcal{I} integrals have been defined in (29) and

$$\begin{aligned} \mathcal{J}^{(A_k)} &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_k \mathcal{A}_k(\mathbf{r}_1, \dots, \mathbf{r}_k) \\ & \quad \times \sum_{m=0}^k F_m(\mathbf{r}_1, \dots, \mathbf{r}_k) (k-2m). \end{aligned} \quad (32)$$

The summation appearing in (32) contains an equal number of positive and negative terms. Thus, it can vanish if the mode

structure is properly chosen. In particular, it vanishes when the F_m 's are invariant with respect to the exchange of the two mode wave functions:

$$F_m = F_{k-m} \Rightarrow \sum_{m=0}^k F_m (k-2m) = 0. \quad (33)$$

This condition is always valid in a ‘‘double-slit’’ BEC interference experiment [8,51–55], where the two modes are identically prepared (within experimental accuracy) and then let to interfere. It is easy to check that the above condition is satisfied whenever $|\psi_a(\mathbf{r})| = |\psi_b(\mathbf{r})|$ at all points. In turn, this is a consequence of the invariance of F_m under local phase transformations $\psi_{a,b}(\mathbf{r}) \rightarrow e^{i\varphi(\mathbf{r})} \psi_{a,b}(\mathbf{r})$.

IV. TYPICALITY OF DENSITY CORRELATIONS

In this section we will specialize the general results obtained for an observable of the form (17) to a particular class, related to the spatial interference of condensates. This is usually accessible to experimentalists and provides an interesting example of typical behavior. We shall analyze the integrated density correlation functions

$$\begin{aligned} \hat{C}_k(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}) &= \int d\mathbf{r} \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{r} + \mathbf{x}_1) \cdots \hat{\rho}(\mathbf{r} + \mathbf{x}_{k-1}) \\ &= \int d\mathbf{r} \hat{G}_k(\mathbf{r}, \mathbf{r} + \mathbf{x}_1, \dots, \mathbf{r} + \mathbf{x}_{k-1}) \\ & \quad + O(N^{k-1}), \end{aligned} \quad (34)$$

where $\hat{\rho}(\mathbf{r}) = \hat{G}_1(\mathbf{r})$. In agreement with Eq. (17), it is clear from the normal-ordered form that the highest-order integral kernel for this class of observables is

$$C_k(\mathbf{r}_1, \dots, \mathbf{r}_k) = \prod_{i=1}^{k-1} \delta(\mathbf{r}_{i+1} - \mathbf{r}_1 - \mathbf{x}_i). \quad (35)$$

The singularities arising in the normal ordering of (34) can be avoided by smearing all densities around the points $(\mathbf{r} + \mathbf{x}_i)$ with functions that take into account the finite experimental spatial resolution.

The \mathcal{I} integrals appearing in the typicality condition (28) can be computed for any couple of mode wave functions using (35):

$$\begin{aligned} \mathcal{I}_{m', m}^{(C_k)} &= \int d\mathbf{r}_1 \cdots d\mathbf{r}_k C_k(\mathbf{r}_1, \dots, \mathbf{r}_k) (\Phi_{m'}^* \Phi_m)(\mathbf{r}_1, \dots, \mathbf{r}_k) \\ &= \int d\mathbf{r} (\Phi_{m'}^* \Phi_m)(\mathbf{r}, \mathbf{r} + \mathbf{x}_1, \dots, \mathbf{r} + \mathbf{x}_{k-1}). \end{aligned} \quad (36)$$

Consider for definiteness $m' > m$. In this case, the function $\Phi_{m'}^* \Phi_m(\mathbf{r}_1, \dots, \mathbf{r}_k)$ is the sum of products of k mode wave functions and k complex conjugates, with the number of ψ_a^* 's exceeding the number of ψ_a 's by $m' - m$. The structure of the products reads

$$\begin{aligned} & \prod_{\sigma=1}^S |\psi_a(\mathbf{r}_{j_\sigma})|^2 \prod_{\tau=1}^T |\psi_b(\mathbf{r}_{j_\tau})|^2 \\ & \quad \times \prod_{\zeta=1}^Z (\psi_a^* \psi_b)(\mathbf{r}_{j_\zeta}) \prod_{\xi=1}^X (\psi_b^* \psi_a)(\mathbf{r}_{j_\xi}), \end{aligned} \quad (37)$$

with $S + T + Z + X = k$, $S + Z = m'$, and $S + X = m$, which implies

$$Z - X = m' - m \quad (38)$$

for all products in $\Phi_{m'}^* \Phi_m$.

A. Plane-wave modes

Let us focus on two modes that are paradigmatic in the description of interference of Bose-Einstein condensates, namely, two counterpropagating plane waves,

$$\psi_a(\mathbf{r}) = e^{ik_0 \cdot \mathbf{r}}, \quad \psi_b(\mathbf{r}) = e^{-ik_0 \cdot \mathbf{r}}. \quad (39)$$

In the next section, we will show how the results obtained in this idealized situation generalize to the more realistic case of two expanding modes that are spatially separated at the initial time.

For the plane-wave modes (39), $|\psi_{a,b}| = 1$ and $\psi_a^* \psi_b = e^{-2ik_0 \cdot \mathbf{r}}$. Inserting these results in the general form of the products (37) and integrating over \mathbf{r} as in (36) yields

$$\prod_{\zeta=1}^Z e^{-2ik_0 \cdot \mathbf{x}_{j_\zeta-1}} \prod_{\xi=1}^X e^{2ik_0 \cdot \mathbf{x}_{j_\xi-1}} \int d\mathbf{r} e^{-2i(m'-m)\mathbf{k}_0 \cdot \mathbf{r}} = 0. \quad (40)$$

Thus, all the contributions to the integral (36) identically vanish for every k and every set of points $(\mathbf{x}_1, \dots, \mathbf{x}_{k-1})$. This implies that condition

$$D_{2k,0}^{(C_k)} = 0 \quad (41)$$

is satisfied by all density correlation functions of the form (34), which are thus typical for $n = o(N)$. The structure of the modes provides interesting information also on the $N^{2k-2}n^2$ part of the fluctuations, arising from (21) and the general expression (30). Since $|\psi_a|$ and $|\psi_b|$ are identically equal to one, the mode wave functions satisfy the symmetry condition (33) for the F_m functions. This implies that the $\mathcal{J}^{(C_k)}$ integrals, defined as in Eq. (32), identically vanish, which, together with the cancellation of the \mathcal{I} integrals, leads to the result

$$D_{2k-2,2}^{(C_k)} = 0. \quad (42)$$

Thus, the highest order of ensemble-dependent contributions to the variance δC_k^2 is indeed $N^{2k-4}n^4$, which should be compared with the order N^{2k-1} of the ensemble-independent fluctuations. This means that if $n = O(N^\alpha)$, the relative fluctuations behave like

$$\frac{\delta C_k}{\overline{C_k}} = \begin{cases} O(\frac{1}{\sqrt{N}}) & \text{for } \alpha \leq 3/4, \\ O((\frac{n}{N})^2) = O(N^{2(\alpha-1)}) & \text{for } \alpha > 3/4. \end{cases} \quad (43)$$

Observe that, when $\alpha \leq 3/4$, the dominant fluctuations are ensemble independent, while they depend on the choice of n in the other case. The transition between the two regimes for a typical observable is generally expected at $\alpha = 1/2$ [see discussion after Eq. (25)]; in this case, however, the transition takes place at the larger threshold value $\alpha = 3/4$ by virtue of the cancellation (42). This behavior, which is general for density correlation functions, generalizes the results obtained in [26] for the Fourier transform of the second-order density

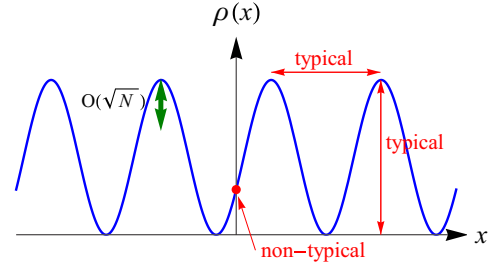


FIG. 1. (Color online) In the asymptotic regime, the outcome of a density measurement in a two-plane-wave mode system is approximated by the function (47). The visibility $\sim O(1)$ and the period $\sim k_0^{-1}$ of the interference pattern are (almost) fixed by typicality, namely, by the typical correlation function (46). The offset ϕ of the pattern, and thus the value of density at the origin, does not appear in the typical observable (46) and fluctuates randomly. The order $O(\sqrt{N})$ of the density fluctuations refers to the case $n = O(N^\alpha)$ with $\alpha \leq 3/4$ [see Eq. (43)].

correlation function

$$\hat{C}_2(\mathbf{x}) = \int d\mathbf{r} \hat{\rho}(\mathbf{r}) \hat{\rho}(\mathbf{r} + \mathbf{x}). \quad (44)$$

We include a brief discussion of this operator to clarify what implications typicality has from an experimental point of view. The expectation value of (44) is generally given by

$$\overline{C_2(\mathbf{x})} = \frac{N^2}{4} \sum_{m=0}^2 \int d\mathbf{r} F_m(\mathbf{r}, \mathbf{r} + \mathbf{x}) + O(N) + O(n^2), \quad (45)$$

which, in the case of plane waves, specializes to

$$\overline{C_2(\mathbf{x})} \simeq N^2 \left[1 + \frac{1}{2} \cos(2\mathbf{k}_0 \cdot \mathbf{x}) \right]. \quad (46)$$

Due to typicality, the function (46) represents the overwhelmingly probable experimental result of a measurement of the observable $\hat{C}_2(\mathbf{x})$, unless $n = O(N)$. This function is the two-point density correlation of a *classical* density (see discussion in Ref. [26])

$$\rho(x) = 2N^2 \cos^2(\mathbf{k}_0 \cdot \mathbf{x} + \phi). \quad (47)$$

Among all the parameters that determine the typical experimental outcome of a density measurement, one, namely, the offset ϕ of the interference pattern, is not determined by typicality since correlation functions do not depend on it. In Fig. 1 a possible outcome of a density measurement in a two-plane-wave mode system is represented, with its typical (and nontypical) features highlighted.

B. Arbitrary expanding modes

Let us now discuss the typicality properties of density correlation functions for a two-mode system that is closer to actual experimental implementations and find out in which cases the results (41) and (42) can be generalized. Let us consider two spatially separated modes $\psi_a(z)$ and $\psi_b(z)$ in one dimension, which are concentrated, respectively, around positions z_a and z_b , with their initial width being much smaller than their distance. Besides these conditions, no other requirement is necessary on the mode wave functions, which

are fully general. The spatial separation implies that the convolution

$$\int dz' \psi_a^*(z') \psi_b(z' + z) =: e^{F_{ab}(z) + i\varphi_{ab}(z)}, \quad (48)$$

with F_{ab} and φ_{ab} being real functions of z , is peaked around a point $z = z_0$, while the convolutions of ψ_a^* with ψ_a and of ψ_b^* with ψ_b are peaked around $z = 0$. If the trapping potential is turned off at the initial time $t = 0$, the particles evolve under the free Hamiltonian $H_0 = -d^2/dz^2$. In the absence of collisions, the k -point functions at $t > 0$ can be obtained by replacing

$$\psi_{a,b}(z) \rightarrow \psi_{a,b}(z,t) = \exp(-itH_0) \psi_{a,b}(z). \quad (49)$$

We are interested in the large-time (*far-field*) regime, in which the evolved wave functions are approximated by the asymptotic form [56]

$$\psi_{a,b}(z,t) \simeq \left(\frac{1}{4\pi it} \right)^{\frac{1}{2}} e^{i\frac{z^2}{4t}} \tilde{\psi}_{a,b} \left(\frac{z}{2t} \right), \quad (50)$$

where $\tilde{\psi}_{a,b}(k) = \int dz e^{-ikz} \psi_{a,b}(z)$ are the Fourier transforms of the *initial* modes. In the far-field approximation, the quantities $\psi_a^* \psi_b$ and their complex conjugates, which enter the typicality condition (28) through products of type (37), are thus related to products of Fourier transforms, which can be expressed through the convolution (48) as

$$\tilde{\psi}_a^*(k) \tilde{\psi}_b(k) = \int dz e^{F_{ab}(z) + i\varphi_{ab}(z) - ikz}. \quad (51)$$

Using a quadratic approximation of F_{ab} around its maximum z_0 , the integral becomes Gaussian, and the product $\psi_a^* \psi_b$ reads

$$\begin{aligned} \psi_a^*(z,t) \psi_b(z,t) &\simeq c(z_0,t) \exp\left(-i\frac{z_0 z}{2t}\right) \\ &\times \exp\left(-\frac{z^2}{8t^2 |F''_{a,b}(z_0)|}\right). \end{aligned} \quad (52)$$

If the spatial period of the complex exponential is much shorter than the standard deviation of the Gaussian part, namely,

$$z_0 \gg \frac{2\pi}{|F''_{a,b}(z_0)|}, \quad (53)$$

the product (52) is consistent with the result for counter-propagating plane-wave modes with time-dependent wave number $k_0(t) = z_0/4t$, modulated by a slowly varying Gaussian envelope, whose width is of the same order as the standard deviation of the far-field-approximated $|\psi_a|^2$ and $|\psi_b|^2$. Thus, the integrals of products (37) with the kernel (35) are exponentially suppressed as $e^{-(z_0 |F''_{ab}|)^2}$. This means that when the relation (53) on the initial wave packets is satisfied,

the typicality condition (41) on correlation functions is fulfilled within a very good level of approximation.

The generalization of the result (42), yielding the cancellation of $N^{2k-2}n^2$ terms in the variance, cannot be extended without further assumptions to the case of expanding modes in the far-field regime. In fact, the relation $|\psi_a(z,t)| = |\psi_b(z,t)|$ might not even be approximately verified in the large-time limit. Condition $\mathcal{J}^{(C_k)}$, which, together with the results discussed above, leads to $D_{2k-2,2}^{(C_k)} = 0$, can be verified only if the low-momentum Fourier components of the two modes are approximately equal [see Eq. (50)]. In the case analyzed in Ref. [26], in which the correlation $C_2(z)$ has been studied for a system with two translated Gaussian modes, it turned out that $D_{2k,2}^{(C_2)} \simeq 0$, as expected from the present general discussion.

V. CONCLUSIONS AND OUTLOOK

In this article we have defined and characterized the typicality of a generic observable in a two-mode Bose system. The results enable one to determine if an observable is typical once the mode wave functions are known and to identify different regimes in the fluctuations around the typical expectation value, as the dimension of the sampled subspace varies. The observable analyzed in the last section is experimentally accessible and can provide a test for typicality.

The identification of typicality criteria for observables helps one understanding which properties are shared by the vast majority of states and which ones have instead wide fluctuations. Remarkably, this distinction is central in determining “good” (macroscopic) observables in both classical and quantum-statistical mechanics [57]. The relation between the results obtained in this article and statistical mechanics will be the object of future research.

It would also be interesting to extend the formalism in order to include more general cases, such as nonuniform samplings or randomly fluctuating modes. Other possible avenues for future investigation will be the analysis of the dynamical effects of typicality [47] and the statistical interpretation of recent experiments on phase randomization in condensates [51–55], as well as the characterization of the typicality of entanglement in a Bose-Einstein condensate [58].

ACKNOWLEDGMENTS

This work was partially supported by Italian Progetti di Rilevante Interesse Nazionale (PRIN) 2010LLKJBX on “Collective quantum phenomena: from strongly correlated systems to quantum simulators” and by the Italian National Group of Mathematical Physics (GNFM-INdAM).

-
- [1] A. S. Parkins and D. F. Walls, *Phys. Rep.* **303**, 1 (1998).
 [2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
 [3] A. J. Leggett, *Rev. Mod. Phys.* **73**, 307 (2001).
 [4] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
 [5] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).

- [6] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, 2008).
 [7] A. J. Leggett, *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems* (Oxford University Press, Oxford, 2006).
 [8] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Science* **275**, 637 (1997).

- [9] E. Altman, E. Demler, and M. D. Lukin, *Phys. Rev. A* **70**, 013603 (2004).
- [10] A. Polkovnikov, E. Altman, and E. Demler, *Proc. Natl. Acad. Sci. USA* **103**, 6125 (2006).
- [11] V. Gritsev, E. Altman, E. Demler, and A. Polkovnikov, *Nat. Phys.* **2**, 705 (2006).
- [12] A. Polkovnikov, *Europhys. Lett.* **78**, 10006 (2007).
- [13] A. Imambekov, V. Gritsev, and E. Demler, in *Ultra-Cold Fermi Gases*, edited by M. Inguscio, W. Ketterle, and C. Salomon, International School of Physics “Enrico Fermi” Vol. 164 (IOS, Amsterdam, 2007), pp. 535–606.
- [14] V. Gritsev, E. Demler, and A. Polkovnikov, *Phys. Rev. A* **78**, 063624 (2008).
- [15] Z. Hadzibabic, S. Stock, B. Battelier, V. Bretin, and J. Dalibard, *Phys. Rev. Lett.* **93**, 180403 (2004).
- [16] S. Stock, Z. Hadzibabic, B. Battelier, M. Cheneau, and J. Dalibard, *Phys. Rev. Lett.* **95**, 190403 (2005).
- [17] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, *Nature (London)* **441**, 1118 (2006).
- [18] P. Krüger, Z. Hadzibabic, and J. Dalibard, *Phys. Rev. Lett.* **99**, 040402 (2007).
- [19] K. Góral, M. Gajda, and K. Rzążewski, *Phys. Rev. A* **66**, 051602(R) (2002).
- [20] Y. Castin and J. Dalibard, *Phys. Rev. A* **55**, 4330 (1997).
- [21] J. Javanainen and S. M. Yoo, *Phys. Rev. Lett.* **76**, 161 (1996).
- [22] J. I. Cirac, C. W. Gardiner, M. Naraschewski, and P. Zoller, *Phys. Rev. A* **54**, R3714(R) (1996).
- [23] T. Wong, M. J. Collett, and D. F. Walls, *Phys. Rev. A* **54**, R3718(R) (1996).
- [24] M. Iazzi and K. Yuasa, *Phys. Rev. A* **83**, 033611 (2011).
- [25] S. Ando, K. Yuasa, and M. Iazzi, *Int. J. Quantum Inf.* **9**, 431 (2011).
- [26] P. Facchi, H. Nakazato, S. Pascazio, F. V. Pepe, and K. Yuasa, *Phys. Rev. A* **89**, 063625 (2014).
- [27] P. Facchi, S. Pascazio, and F. V. Pepe, [arXiv:1501.05881](https://arxiv.org/abs/1501.05881).
- [28] M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs Vol. 89 (American Mathematical Society, Providence, RI, 2001).
- [29] P. Hayden, D. W. Leung, and A. Winter, *Commun. Math. Phys.* **265**, 95 (2006).
- [30] P. Facchi, U. Marzolino, G. Parisi, S. Pascazio, and A. Scardicchio, *Phys. Rev. Lett.* **101**, 050502 (2008).
- [31] A. De Pasquale, P. Facchi, V. Giovannetti, G. Parisi, S. Pascazio, and A. Scardicchio, *J. Phys. A* **45**, 015308 (2012).
- [32] P. Facchi, G. Florio, G. Parisi, S. Pascazio, and K. Yuasa, *Phys. Rev. A* **87**, 052324 (2013).
- [33] F. D. Cunden, P. Facchi, and G. Florio, *J. Phys. A* **46**, 315306 (2013).
- [34] S. Lloyd, Ph.D. thesis, Rockefeller University, 1991.
- [35] H. Tasaki, *Phys. Rev. Lett.* **80**, 1373 (1998).
- [36] J. Gemmer and G. Mahler, *Eur. Phys. J. B* **31**, 249 (2003).
- [37] S. Popescu, A. J. Short, and A. Winter, [arXiv:quant-ph/0511225](https://arxiv.org/abs/quant-ph/0511225).
- [38] S. Popescu, A. J. Short, and A. Winter, *Nat. Phys.* **2**, 754 (2006).
- [39] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, *Phys. Rev. Lett.* **96**, 050403 (2006).
- [40] A. Sugita, *Nonlinear Phenom. Complex Syst.* **10**, 192 (2007).
- [41] P. Reimann, *Phys. Rev. Lett.* **99**, 160404 (2007); **101**, 190403 (2008).
- [42] M. Rigol, V. Dunjko, and M. Olshanii, *Nature (London)* **452**, 854 (2008).
- [43] J. Cho and M. S. Kim, *Phys. Rev. Lett.* **104**, 170402 (2010).
- [44] S. Sugiura and A. Shimizu, *Phys. Rev. Lett.* **108**, 240401 (2012); **111**, 010401 (2013).
- [45] G. S. Paraoanu, *J. Low. Temp. Phys.* **153**, 285 (2008); *Phys. Rev. A* **77**, 041605(R) (2008).
- [46] K. Maussang, G. E. Marti, T. Schneider, P. Treutlein, Y. Li, A. Sinatra, R. Long, J. Estève, and J. Reichel, *Phys. Rev. Lett.* **105**, 080403 (2010).
- [47] P. Facchi, H. Nakazato, S. Pascazio, F. V. Pepe, G. A. Sekh, and K. Yuasa, *Int. J. Quantum Inf.* **12**, 1560019 (2014).
- [48] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
- [49] We shall focus on this case for simplicity, but the formalism, as well as the results, can be generalized to kernels involving derivatives. On the other hand, momentum-dependent observables can be more easily treated by using correlation functions in momentum space, constructed with the Fourier-transformed field operators $\hat{\Psi}(\mathbf{p}) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \hat{\Psi}(\mathbf{r})$.
- [50] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996), Vol. 1.
- [51] T. Schumm, S. Hofferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, *Nat. Phys.* **1**, 57 (2005).
- [52] S. Hofferberth, I. Lesanovsky, B. Fischer, J. Verdu, and J. Schmiedmayer, *Nat. Phys.* **2**, 710 (2006).
- [53] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, *Nature (London)* **449**, 324 (2007).
- [54] S. Hofferberth, I. Lesanovsky, T. Schumm, A. Imambekov, V. Gritsev, E. Demler, and J. Schmiedmayer, *Nat. Phys.* **4**, 489 (2008).
- [55] T. Langen, R. Geiger, M. Kuhnert, B. Rauer, and J. Schmiedmayer, *Nat. Phys.* **9**, 640 (2013).
- [56] G. Teschl, *Mathematical Methods in Quantum Mechanics* (American Mathematical Society, Providence, RI, 2009).
- [57] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1* (Pergamon, Oxford, 1980).
- [58] F. Benatti, R. Floreanini, and U. Marzolino, *Phys. Rev. A* **85**, 042329 (2012).