Thou shalt not say *"at random"* in vain: Bertrand's paradox exposed

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Abstract

We review the well known Bertrand paradoxes, and we first maintain that they do not point to any probabilistic inconsistency, but rather to the risks incurred with a careless use of the locution *at random*. We claim then that these paradoxes spring up also in the discussion of the celebrated Buffon's needle problem, and that they are essentially related to the definition of (geometrical) probabilities on *uncountably* infinite sets. A few empirical remarks are finally added to underline the difference between *passive* and *active* randomness, and the prospects of any experimental decision

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1 The Bertrand paradoxes

In the first chapter of his classic treatise [1] Joseph Bertrand dwells for a while on the definition of probability, and in particular – in the paragraphs 4-7 – he remarks that the random models with an *infinite* number of possible results are prone to particularly insidious misunderstandings[1]. He lists then a few examples of problems each admitting equally legitimate, but contradictory answers and suggests then that our questions are *ill posed*, or more precisely that the required probabilities, based on some *at random* (*au hasard*) choice, "sont impossibles à assigner si la question n'est pas précisée davantage" (see [1] p. 7). How it will be made clear later, however – and

[1] "L'infini n'est pas un nombre; on ne doit pas, sans explication, l'introduire dans les raisonnements ... Choisir *au hasard* entre un nombre infini de cas possibles, n'est pas une indication suffisante." See [1] p. 4

how it was likely clear to Bertrand himself – the crucial point is less the infinity of the possible outcomes, than their *uncountable* infinity: a feature shared with other time honored problems, as for instance that of *Buffon's needle* also discussed later in the present paper. There is room to argue indeed that even for *countably* infinite sample spaces the paradoxes do not arise because the very notion of *at random* – as long as it is associated with some idea either of *equiprobability* or of *uniformity* – that has not there a straight extension, can be asymptotically retrieved in a way free from ambiguities. Bertrand on the other hand, while correctly pointing out that the questions proposed in his examples are fallacious exactly because our use of the said locution is too careless, fails to elaborate further on this point leaving the reader with the odd feeling that something could be inconsistent in the general notion of randomness. A negligence extended – with few notable exceptions [2, 3] – also to many of the modern textbooks that still bother to mention this topic[2]

The aim of the present paper is then to address this very point: what are the root and the scope of these seeming inconsistencies? And in accomplishing our task we will linger first in the Section 2 on the example that is widely acknowledged today as the paradigmatic Bertrand paradox because its results look especially puzzling. We will then proceed in Section 3 to extend similar remarks to the Buffon needle problem, and in Section 4 to argue that while the paradoxes certainly arise in the event of (geometrical) probabilities defined on uncountably infinite sets, asymptotically equiprobable countably infinite sets (as for instance the rational numbers discussed in the Appendix A) seem to share the fate of finite sets in avoiding these ambiguities. In the last Section 5 we will finally conclude by adding a few remarks about the meaning of a possible experimental discrimination among the different legitimate solutions

2 The circle, the triangle and the chord

Usually the problem is proposed in the following way: looking at the Figure 1, take *at random* a chord on the circle Γ of radius 1: what is the probability that its length will exceed that of the edge of an inscribed equilateral triangle (namely will exceed $\sqrt{3}$)? Three acceptable solutions are possible, but their answers are all *numerically* different (we always make reference to the Figure 1):

1. To take a chord at random is equivalent to choose the location of its middle point (its orientation would be an aftermath), and to get the chord longer than the triangle edge it is necessary and sufficient to take this middle point inside the concentric circle γ with radius $^1/_2$ inscribed in the triangle. The required

[2]See for instance [4] whose final remarks (p. 9) are not really helpful: "We have thus found not one but three different solutions for the same problem! One might remark that these solutions correspond to three different experiments. This is true but not obvious and, in any case, it demonstrates the ambiguities associated with the classical definition, and the need for a clear specification of the outcomes of an experiment and the meaning of the terms 'possible' and 'favorable' "

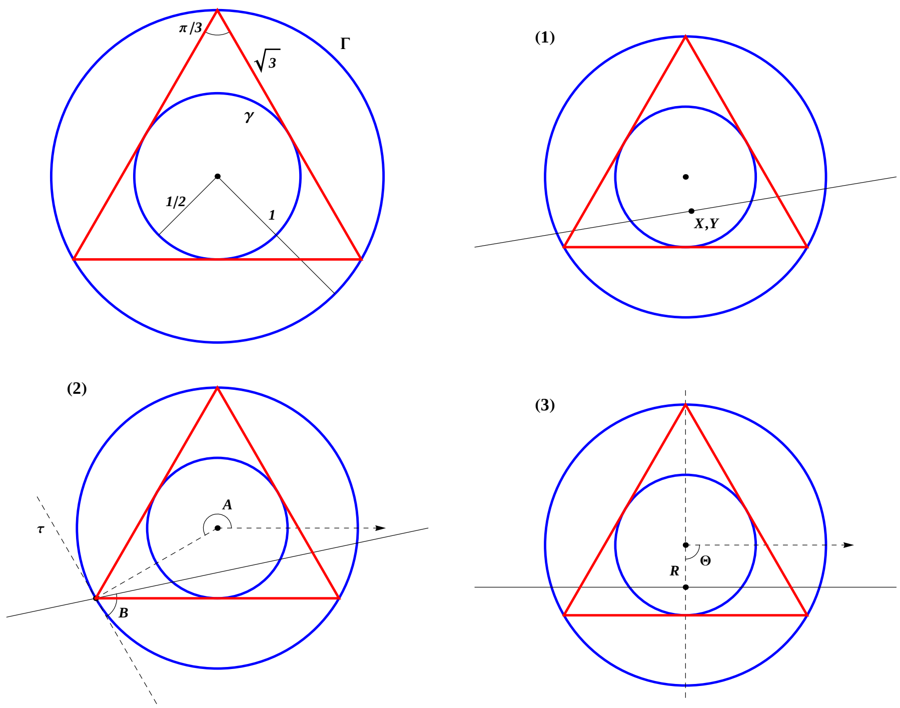

Figure 1: Bertrand's paradox

probability is then the ratio between the area $^{\pi}/_{4}$ of γ and the area π of Γ, and consequently we have $p_1 = {}^{1}/_{4}$

2. By symmetry the position of one chord endpoint along the circle is immaterial to our calculations: then, for a given endpoint, the chord length will only be contingent on the angle (between 0 and π) with the tangent line τ in the chosen endpoint. If then we draw the triangle with one vertex in the chosen endpoint, the chord at random will exceed its edge if the angle with the tangent falls between $^{\pi}/_{3}$ and $^{2\pi}/_{3}$, and the corresponding probability will be $p_2 = {}^{1}/_{3}$

3. Always by symmetry, the random chord direction does not affect the required probability. Fix then such a direction, and remark that the chord will exceed $\sqrt{3}$ if its intersection with the orthogonal diameter falls within a distance from the center smaller than $^{1}/_{2}$: this happens with probability $p_3 = {}^{1}/_{2}$

We are then left with three different ($^{1}/_{4}$, $^{1}/_{3}$ and $^{1}/_{2}$), but equally acceptable answers. To find the paradox origin we must remember that taking a number *at random* in an uncountably infinite domain usually means that this number is there *uniformly* distributed. It is possible to show however (as also hinted in [2]) that what is considered as uniformly distributed in every single proposed solution can not at the same time be uniformly distributed in the other two: in other words,

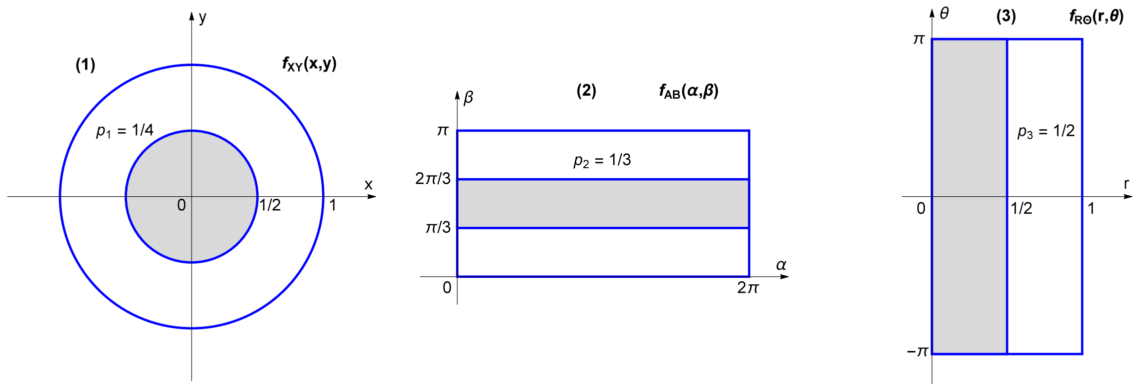

Figure 2: Supports of the uniform *pdf*'s (1), (2) and (3): the shaded areas correspond to the three Bertrand probabilities p_1, p_2 and p_3

in our three solutions – by differently choosing *what* is uniformly distributed – we surreptitiously adopt three different probability distributions, and consequently it is not astonishing at all that the three answers mutually disagree

To be more precise, let us define (see Figure 1) the three *rv* (random variable) pairs representing the coordinates describing the position of our chord in the three proposed solutions:

1. the Cartesian coordinates (X, Y) of the chord middle point
2. the angles (A, B) respectively giving the position of the fixed endpoint and the chord orientation w.r.t. the tangent
3. the polar coordinates (R, Θ) of the chord-diameter intersection

In every instance however we apparently make the concealed (namely not explicitly acknowledged) hypothesis that the corresponding pair of coordinates is uniformly distributed, but these three assumptions are not mutually consistent, as we will see at once, because they require three different probability measures on the probability space where all our *rv*'s are defined. In particular, and by adopting the notation

$$\chi_{[a,b]}(x) = \begin{cases} 1, & \text{if } a \leq x \leq b; \\ 0, & \text{else} \end{cases}$$

the three solutions respectively assume the following uniform, joint distributions (see also Figure 2 for a graphical account of their respective supports):

1. the joint, uniform *pdf* on the unit circle in $\boldsymbol{R}^2$

$$f_{XY}(x.y) = \frac{1}{\pi}\,\chi_{[0,1]}(x^2 + y^2) \tag{1}$$

   for the pair (X, Y): here the two *rv*'s X and Y are *not* independent

2. the joint, uniform *pdf* on the rectangle $[0, 2\pi] \times [0, \pi]$ in $\boldsymbol{R}^2$

$$f_{AB}(\alpha, \beta) = \frac{1}{2\pi^2}\,\chi_{[0,2\pi]}(\alpha)\chi_{[0,\pi]}(\beta) \tag{2}$$

for the pair (A, B) with independent components

3. and finally the joint, uniform *pdf* on the rectangle $[0, 1] \times [-\pi, \pi]$ in $\boldsymbol{R}^2$

$$f_{R\Theta}(r, \theta) = \frac{1}{2\pi}\,\chi_{[0,1]}(r)\chi_{[-\pi,\pi]}(\theta) \tag{3}$$

for the pair (R, Θ) again with independent components

Surely enough, if we would adopt a unique probability space for all of our three solutions, the three numerical results would be exactly coincident, but in this case only one of the three *rv* pairs could possibly be uniformly distributed, while the other joint distributions should be derived by adopting the well known procedures established for the functions of *rv*'s (see for instance [4], Sections 5.2, 6.2 and 6.3). The crucial point here is that there are in fact some precise transformations allowing the change from a pair of *rv*'s to the other: by using these transformations we can show indeed that if a pair is jointly uniform, then the other two can not have the same property

Without going into the details of every possible combination in our problem, we will confine ourselves to discuss just the relations between the solutions (1) and (3). The transformations between the Cartesian coordinates (X, Y) and the polar ones (R, Θ) are well known:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} & 0 < r \\ \theta = \arctan {}^{y}\!/_{x} & -\pi < \theta \leq \pi \end{cases}$$

with a Jacobian determinant

$$J(r, \theta) = \begin{vmatrix} {}^{\partial r}\!/_{\partial x} & {}^{\partial r}\!/_{\partial y} \\ {}^{\partial\theta}\!/_{\partial x} & {}^{\partial\theta}\!/_{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -{}^{1}\!/_{r}\,\sin\theta & {}^{1}\!/_{r}\,\cos\theta \end{vmatrix} = \frac{1}{r}$$

As a consequence (see for instance [4] Section 6.3), if (X, Y) have the joint uniform *pdf* (1), then the pair (R, Θ) will not be uniform and will have instead the *pdf*

$$f^{(1)}_{R\Theta}(r, \theta) = \frac{r}{\pi}\,\chi_{[0,1]}(r)\chi_{[-\pi,\pi]}(\theta)$$

which is apparently different from the $f_{R\Theta}$ in (3). By taking advantage of this new distribution $f^{(1)}_{R\Theta}$ (coherent now with the choice of a jointly uniform pair X, Y) it is easy to see that the required probability within the framework of the solution (3) would be

$$p_3^{(1)} = \int_0^{\frac{1}{2}} \frac{r}{\pi}\,dr \int_{-\pi}^{\pi} d\theta = \frac{1}{4}$$

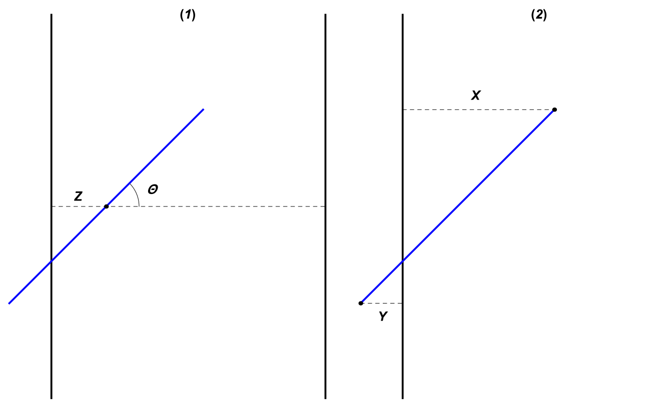

Figure 3: Bertrand's paradox for Buffon's needles

instead of $p_3 = {}^1/_2$, in perfect agreement with the value $p_1 = {}^1/_4$ of the solution (1). Hence the paradox ghosts would daunt us just as long as we unwittingly suppose that in our three solutions the coordinates can all be *at once* uniformly distributed (hiding that under the careless locution *at random*), and they will disappear instead as soon as we consistently adopt a unique probability space for all our *rv*'s

3 Bertrand *vs* Buffon

It is interesting to remark now that, while it is known that by the turn of the century several different solutions of the Bertrand question were added[3] to the usual three recalled in the previous section, nobody at our knowledge seem to have noticed that the same kind of paradoxes does in fact appear also in the discussion of the celebrated Buffon needle problem. In its simplest version[4] a needle of unit length is thrown *at random* on a table where a few parallel lines are drawn at a unit distance: what is the probability that the needle will lie across one of these lines? In the classical answer to this question, since the lines are drawn periodically on the table, it will be enough to study the problem with only two lines by supposing that the needle center does fall between them. The position of the said center along the direction of the parallel lines is also immaterial. The needle position is then defined by just two rv's: the distance Z of its center from the left line, and the angle Θ between the needle and a perpendicular to the parallel lines (see (1) in Figure 3). That the needle is thrown *at random* here means that the pair of rv's Θ, Z is uniform in $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 1]$, namely that their joint *pdf* is

$$f_{\Theta Z}(\theta, z) = \frac{1}{\pi}\, \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\theta)\, \chi_{[0,1]}(z) \tag{4}$$

[3]See for instance [5] quoted in [3]

[4]For a more complete discussion see for instance [2] and [6]

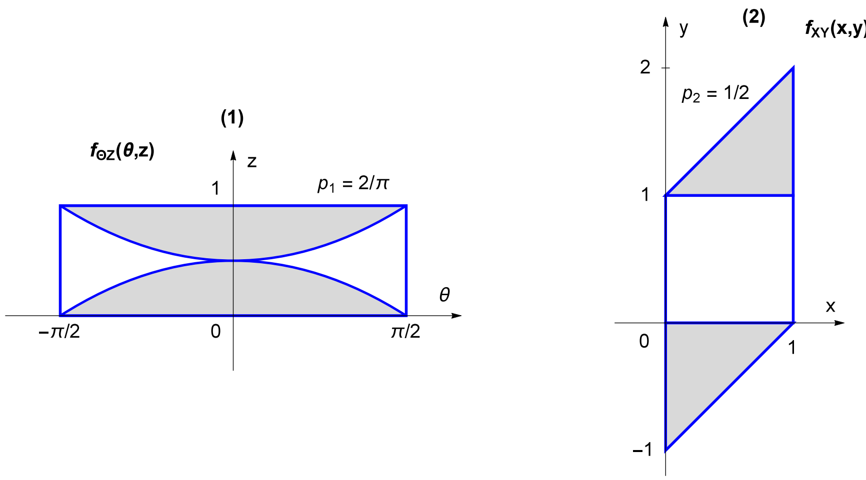

Figure 4: Supports of the uniform *pdf*'s (4) and (5): the shaded areas correspond to the two Buffon probabilities p_1 and p_2

while, with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the needle will lie across a line either when $x \leq {}^1\!/_2 \cos\theta$, or when $x \geq 1 - {}^1\!/_2 \cos\theta$ (see again (1) in Figure 3). Just by inspecting the *pdf* (1) in Figure 4 it is then easy to find out that the required probability is

$$p_1 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{2} \, d\theta = \frac{2}{\pi}$$

In the spirit of the Bertrand paradoxes, however, we can give a different answer to the Buffon question (see (2) in Figure 3): the needle position is now identified by looking first to its (vertically) upper end, and by recording its distance X from the left line. Then we consider where its other (lower) end falls and we mark down its distance Y from the same left line. If the said left line is in the origin of a horizontal axis, it is apparent that for every value $0 \leq x \leq 1$ of X, the possible values of Y will be between $x - 1$ and $x + 1$ (because apparently $|x - y| \leq 1$), and in this framework to throw the needle *at random* will mean that the joint distribution of X, Y is uniform in the domain shown in (2) of the Figure 4, namely

$$f_{XY}(x, y) = \frac{1}{2} \chi_{[0,1]}(x) \, \chi_{[0,1]}(|x - y|) \tag{5}$$

On the other hand it is apparent that for every $0 \leq x \leq 1$ the needle will cross a line when either $x - 1 \leq y \leq 0$, or $0 \leq y \leq x + 1$, so that the required probability will correspond to the shaded area in (2) of Figure 4 and hence now $p_2 = {}^1\!/_2$. These remarks show then that also the Buffon needle problem is not impervious to paradoxes, and this could be more than just a trifle because of its peculiar *experimental* status, as will be argued in the subsequent Section 5

4 Counting and measuring

Since the Bertrand paradoxes arise from a careless use of the locution *taking at random*, it would expedient to recall once more that this is not understood here as the drawing of some outcome ω out of a set Ω according to some arbitrary probability measure, but stands rather for assuming that there is no reason to think that there are preferred outcomes $\omega \in \Omega$, these being supposed instead to be equally likely. For sets of numbers this kind of randomness is enforced either by sheer *equiprobability* (on the finite sets, by *counting*), or by distribution *uniformity* (on the bounded, Lebesgue measurable, uncountable sets, by *measuring*). On the other hand infinite, countable sets and unbounded, uncountable sets are both excluded from these egalitarian probability attributions because their elements can be made neither equiprobable (with a non vanishing probability), nor uniformly distributed (with a non vanishing probability density). This in particular seems to point to the fact that infinite, countable sets should be barred even from discussing the Bertrand problems because, for instance, we can not make all the natural numbers equiprobable with a non zero probability (neither by counting, nor by measuring), while the paradoxes are essentially based on a misunderstanding about this kind of randomness. In these cases however it is possible to start with some proper (not equiprobable) probability distribution, and then to make them ever closer – in a suitable, approximate sense – to an equiprobable one: we will then speak of *asymptotic equiprobability* (see Appendix A and [7] for more details). This notion will be used here to argue that the Bertrand paradoxes arises exclusively in connection to *uncountable* infinite sets of numbers, but neither for finite, nor for countably infinite ones

To clarify this last point it would be expedient to consider another, more simple case among the Bertrand examples (see [1] p. 4): if we ask what is the probability that a *real number* x chosen at random between 0 and 100 is larger than 50, our natural answer is $^1/_2$. Since however the real numbers between 0 and 100 are also bijectively associated to their squares between 0 and 10 000, we also instinctively feel that our question should be equivalent to ask for the probability that the square of a real random number turns out to be larger than $50^2 = 2\,500$. When however we think of taking this last number *at random* between 0 and 10 000, instinctively again we are inclined to answer that the probability of exceeding 2 500 should now be $^3/_4$ instead of $^1/_2$. The two problems look equivalent, but their two intuitive answers (apparently both legitimate) are different

Predictably the paradox resolution is similar to that of Section 2: we would readily concede that the probability to exceed 50 for a real number X taken *at random* in $[0, 100]$ is $p_1 = {}^1/_2$. When however we ask for the probability that X^2 taken *at random* in $[0, 10\,000]$ exceeds $50^2 = 2\,500$, we surreptitiously change our measure by supposing that now X^2 is uniform in $[0, 10\,000]$ and we find $p_2 = {}^3/_4$. But the fact is – as in the previous example – that if X is uniform in $[0, 100]$, then X^2 can not be uniform in $[0, 10\,000]$, and vice-versa. More precisely, if the *pdf* of X

is the uniform

$$f_X(x) = \frac{1}{100}\begin{cases} 1 & \text{for } 0 \le x \le 100, \\ 0 & \text{else} \end{cases}$$

then the corresponding, non uniform *pdf* of $Y = X^2$ is (see again [4] Section 5.2)

$$f_Y(y) = \frac{1}{200\sqrt{y}}\begin{cases} 1 & \text{for } 0 \le y \le 10\,000, \\ 0 & \text{else} \end{cases}$$

and of course the paradox disappears because now, in agreement with $p_1 = {}^1/_2$, we would have

$$p_2^{(1)} = \int_0^{2\,500} f_Y(y)\,dy = \frac{1}{200}\int_0^{2\,500}\frac{dy}{\sqrt{y}} = \frac{1}{2}$$

It is easy to see moreover that the paradox does not show up at all when we consider the *finite* version of this problem: if we ask for the probability ($p_1 = {}^1/_2$) of choosing *at random* an *integer number* n larger than 50 among the (*equiprobable*) integer numbers from 1 to 100, we would in fact recover the same answer ($p_2 = {}^1/_2$) also by asking to calculate the probability of choosing *at random* a number larger than 2 500 among the squared integers $1, 4, 9, \ldots, 10\,000$, because now our set is again constituted of just 100 equiprobable elements and there are 50 larger than 2 500. The crucial difference with the previous continuous version of the problem is that in the case of finitely many (equiprobable) possible results we just *enumerate* the *favorable* and the *possible* items (a situation not changed by squaring the numbers), while for the continuous real numbers (geometric probabilities) we *compare* the length of the intervals: all is contingent indeed on the difference between *counting* and *measuring*

This situation, albeit trickier, is not essentially changed for *countably infinite* possible outcomes, but for the fact that in this case they can not be made strictly equiprobable by direct enumeration. Take for instance the problem of asking for the probability of choosing *at random* a *rational number* $q = {}^n/_m$ larger than 50 among the rational numbers in $[0, 100]$ that are famously an infinite, countable set everywhere dense among the real numbers. It is shown in the Appendix A that – in agreement with our intuition – this probability tends to $p_1 = {}^1/_2$ when, going around the problem of actually enumerating them, the rational numbers in $[0, 100]$ are made asymptotically equiprobable. If however we subsequently ask to calculate the probability of choosing *at random* a number larger than 2 500 among the squared rational numbers $q^2 = {}^{n^2}/_{m^2}$ in $[0, 10\,000]$ we in fact recover the same answer $p_2 = {}^1/_2$ with no possible ambiguity because now – in a way recalling the case of integer numbers – we must make asymptotically equiprobable not all the rational numbers in $[0, 10\,000]$, but rather only those that are squares of rational numbers. Not every rational q is indeed a squared rational, and the (tiny, but non zero) probability of $q^2 = {}^{n^2}/_{m^2}$ being larger than 2 500 exactly coincides with that of $q = {}^n/_m$ being larger than 50: there is no possible sharing of probability with the infinitely many other (non squared) rationals that in any case would never show up in the process of drawing at random squared rationals

The critical feature – common to both the finite and the countably infinite cases – appears here to be the possibility that every single element be endowed with its own individual *non vanishing* probability that it also carries with him in every conceivable one-to-one transformation: no amount of probability must indeed be shared with numerical results other than the transformed results, and the distribution remain the same in the transformed sample space. A situation totally at variance with that of a *geometrical* probability on a uncountably infinite set where every sample is usually entitled only to strictly zero probability and the transformations usually imply a stretching of the probability measure

5 An almost empiric conclusion

Going back to the example discussed in the Section 2, we have argued that the paradoxes "disappear as soon as we consistently adopt a unique probability space for all our *rv*'s", so that – paradoxes notwithstanding – there are no possible formal inconsistencies within our overall probabilistic framework. But this is sheer mathematics, and we are left anyway with three (or more) possible, coherent and perfectly legitimate, probabilistic models giving rise to three numerically different results: which one is *true*, in the sense that it corresponds to the *physical reality*? This problem of course can not be solved with a calculation, and should instead be settled – if possible – by comparing the solutions proposed with some empiric result. In other words one should simply *perform the experiment of choosing at random a chord on a circle* (or something equivalent) in order to compare then its statistics with the calculations: something that to date, at our knowledge, has not yet been done once and for all, and it is not even exactly tackled in recent contributions [8] where, for all the emphasis on solving this hard part of the paradox, the discussion seems to be essentially restricted to the mathematical models

In this vein we will add here just a few final remarks: first, the possible experiment can not definitely be a *simulated* one performed on a computer. In this case indeed our experimenter should *a priori* choose one of the three models to program his computer to produce a particular pair of uniformly distributed coordinates: but in so doing he would have already decided the outcomes of the experiment that coherently will now confirm the chosen model: this apparently will prove nothing. Second, it is possible that even in some real, physical experiment the outcome can be influenced by the choice of what exactly we decide to measure [8]: different experimental settings could point to different facets of the physical reality, and after all the probability is not listed among the concrete things of this world (see for instance the unconventional viewpoint displayed in [9], vol. 1, *Preface*) representing rather the state of our information. Third, the previous remark also lays bare the difference between experiments where we study a *passive randomness* produced by an independent external world (think for instance to the statistics of the usual empirical measures, or even to quantum mechanics: randomness is there completely outside of our control and show characters that must be discovered rather than produced), in contrast with an *active randomness* where we try to produce some kind of previously planned events, namely to empirically enforce some idea of randomness as in both the Bertrand paradox and the Buffon needle, and from a different standpoint in every computer simulation. These two sorts of randomness appear epistemologically rather different and this dissimilarity could be well worth of further inquiry

We can not refrain however from pointing out in the end that the presence of Bertrand-type paradoxes even in the discussion of Buffon's needle sheds a different light on this problem: it is known indeed that the classical calculation of the Buffon needle, giving the probability $p_1 = {}^2/_\pi$, also stimulated several experiments used to get an empirical determination of the approximate value of the number π (four of these tests dated from 1850 to 1901 are listed for instance in [2]), and famously known as pioneering examples of the *Monte Carlo* method. Despite a few reservations about the reliability of these results [2], it is striking that all these four experiments point to a number in the range between 3.14 and 3.16, while our second, proposed alternative solution with $p_2 = {}^1/_2$ would require results clustering around 4.00. It is possible – as suggested above – that the quoted results are biased by some unaware bent toward π, but if confirmed they would suggest that there could be an empirical meaning in the locution *at random* because, at least in the case of Buffon's needle, the experiments appear to be able to favor one among several formally legitimate solutions. But it is also apparent by now that a possible answer to these questions would lie outside the reach of this paper, so that for the time being we will stay content with having just clarified the meaning and the scope of the Bertrand paradoxes and their link to the Buffon needle, by leaving to future inquiries the practical task of empirically deciding among the mathematical models

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A Taking rational numbers at random

Rational numbers are famously countable, and hence they can be put in a sequence. Since however they are a dense subset of the real numbers, every rational number is a cluster point, and hence no sequence encompassing all of them can ever converge, not to say be monotone. In any case their countability certainly allows the allotment of discrete distributions with non vanishing probabilities for every item: since they are infinite, however, they can never be exactly *equiprobable*. In this appendix (further details are available in [7]) we will outline a procedure to give distributions on the rationals in $[0, 1]$, a set that we will shortly denote as $\mathbb{Q}_0 = \mathbb{Q} \cap [0, 1]$, and we will investigate if and how they can be made *asymptotically equiprobable*

It is however advisable to assert right away that the distribution of a *rv* Q taking values in $\mathbb{Q}_0$ must anyhow be of a discrete type, allotting (possibly non vanishing) probabilities to the individual rational numbers $q \in \mathbb{Q}_0$: conceivable continuous set

	n									
m	0	1	2	3	4	5	6	7	8	...
1	0	1								
2	0	$^1/_2$	1							
3	0	$^1/_3$	$^2/_3$	1						
4	0	$^1/_4$	$^2/_4$	$^3/_4$	1					
5	0	$^1/_5$	$^2/_5$	$^3/_5$	$^4/_5$	1				
6	0	$^1/_6$	$^2/_6$	$^3/_6$	$^4/_6$	$^5/_6$	1			
7	0	$^1/_7$	$^2/_7$	$^3/_7$	$^4/_7$	$^5/_7$	$^6/_7$	1		
8	0	$^1/_8$	$^2/_8$	$^3/_8$	$^4/_8$	$^5/_8$	$^6/_8$	$^7/_8$	1	
$\vdots$	$\vdots$									$\ddots$

Table 1: Table of rational numbers $q = {}^n/_m$ with repetitions: many fractions are reducible to canonical forms already present in earlier positions

functions – namely with continuous, albeit perhaps not absolutely continuous, *cdf* (cumulative distribution function) – would turn out to be not countably additive, and hence would not qualify as measures, not to say as probability distributions. Every continuous *cdf* for Q would indeed entail that at the same time $\boldsymbol{P}\{Q = q\} = 0,\ \forall q \in \mathbb{Q}_0$, and $\boldsymbol{P}\{Q \in \mathbb{Q}_0\} = 1$, while $\mathbb{Q}_0$ apparently is the countable union of the disjoint, negligible sets $\{q\}$: in plain conflict with the countable additivity. This in particular rules out for the numbers in $\mathbb{Q}_0$ the possibility of being *uniformly distributed* (an imaginable surrogate of equiprobability evoked by the density of the rational numbers): this property would in fact require a continuous *cdf*

Taking advantage now of the well known diagram used to prove the countability of the rational numbers, we will consider two dependent *rv*'s M and N with integer values

$$m = 1, 2, \dots \qquad\qquad n = 0, 1, 2, \dots, m$$

and acting respectively as denominator and numerator of the random rational number $Q = {}^N/_M \in [0, 1]$. As a consequence Q will take the values $q = {}^n/_m$ arrayed in a triangular scheme as in Table 1. It is apparent however that in this way every rational number q shows up infinitely many times due to the presence of reducible fractions: for instance – with the usual notation for repeating decimals – we have

$$0.5 = {}^1/_2 = {}^2/_4 = {}^3/_6 = \dots \qquad 0.\overline{3} = {}^1/_3 = {}^2/_6 = \dots \qquad 0.75 = {}^3/_4 = {}^6/_8 = \dots$$

If we adopt however the notation

$$q \doteq {}^n/_m$$

to indicate that ${}^n/_m$ is the unique irreducible representation of a rational number q, namely that n and m are co-primes, the previous examples will be listed as

$$0.5 \doteq {}^1/_2 \qquad 0.\overline{3} \doteq {}^1/_3 \qquad 0.75 \doteq {}^3/_4$$

By introducing now a joint distributions of N and M

$$\begin{aligned} &\boldsymbol{P}\{M=m\} && m=1,2,\dots \\ &\boldsymbol{P}\{N=n\,|M=m\} && n=0,1,2,\dots,m \\ &\boldsymbol{P}\{N=n,M=m\}=\boldsymbol{P}\{N=n\,|M=m\}\;\boldsymbol{P}\{M=m\} \end{aligned}$$

we will have for every rational $0\le q\doteq {}^{n}/_{m}\le 1$ the discrete distribution

$$\begin{aligned} \boldsymbol{P}\{Q=q\} &= \sum_{\ell=1}^{\infty}\boldsymbol{P}\{N=\ell n,M=\ell m\} \\ &= \sum_{\ell=1}^{\infty}\boldsymbol{P}\{N=\ell n\,|M=\ell m\}\;\boldsymbol{P}\{M=\ell m\} \end{aligned} \tag{6}$$

This also allows to define the *cdf* of Q as (here of course $x\in\boldsymbol{R}$)

$$\begin{aligned} F_Q(x) &= \boldsymbol{P}\{Q\le x\}=\boldsymbol{P}\{N\le Mx\}=\sum_{m=1}^{\infty}\boldsymbol{P}\{N\le mx\,|M=m\}\,\boldsymbol{P}\{M=m\} \\ &= \sum_{m=1}^{\infty}F_N(mx|M=m)\boldsymbol{P}\{M=m\} \end{aligned} \tag{7}$$

and hence also the probability of Q falling in $(a,b]$ for $0\le a<b\le 1$ real numbers:

$$\begin{aligned} \boldsymbol{P}\{a<Q\le b\} &= F_Q(b)-F_Q(a) \\ &= \sum_{m=1}^{\infty}\big[F_N(mb|M=m)-F_N(ma|M=m)\big]\boldsymbol{P}\{M=m\} \end{aligned} \tag{8}$$

Notice that the conditional *cdf* of N can also be given as

$$\begin{aligned} F_N(x|M=m) &= \boldsymbol{P}\{N\le x\,|M=m\}=\sum_{n=0}^{m}\boldsymbol{P}\{N=n\,|M=m\}\,\vartheta(x-n) \\ &= \sum_{n=0}^{\lfloor x\rfloor}\boldsymbol{P}\{N=n\,|M=m\} \end{aligned} \tag{9}$$

where

$$\vartheta(x)=\begin{cases} 1 & x\ge 0 \\ 0 & x<0 \end{cases}$$

is the Heaviside function, while for every real number x, the symbol $\lfloor x\rfloor$ denotes the *floor* of x, namely the greatest integer less than or equal to x. As a consequence the equations (7) and (8) also take the form

$$F_Q(x) = \sum_{m=1}^{\infty}\boldsymbol{P}\{M=m\}\sum_{n=0}^{\lfloor mx\rfloor}\boldsymbol{P}\{N=n\,|M=m\} \tag{10}$$

$$\boldsymbol{P}\{a<Q\le b\} = \sum_{m=1}^{\infty}\boldsymbol{P}\{M=m\}\,(1-\delta_{\lfloor ma\rfloor,\lfloor mb\rfloor})\sum_{n=\lfloor ma\rfloor+1}^{\lfloor mb\rfloor}\boldsymbol{P}\{N=n\,|M=m\} \tag{11}$$

where the Kronecker delta takes into account the fact that when $\lfloor mb \rfloor = \lfloor ma \rfloor$ the difference vanishes, so that $\lfloor mb \rfloor \geq \lfloor ma \rfloor + 1$.

Let us suppose now for simplicity that for a given denominator $m \geq 1$ the $m+1$ possible values of the numerator $n = 0, 1, \ldots, m$ are equiprobable in the sense that

$$\boldsymbol{P}\{N = n \,|M = m\} = \frac{1}{m+1} \qquad\qquad n = 0, 1, \ldots, m$$

In this case for the distribution, with n, m co-primes and $0 \leq n \leq m$, from (6) we have

$$\boldsymbol{P}\{Q = q\} = \sum_{\ell=1}^{\infty} \frac{\boldsymbol{P}\{M = \ell m\}}{\ell m + 1} \qquad\qquad q \doteq {}^{n}/_{m} \tag{12}$$

while for the *cdf* (10) we have from (9)

$$\begin{aligned} F_N\left(mx\,|M = m\right) &= \frac{1}{m+1}\sum_{n=0}^{m} \vartheta\left(mx - n\right) = \begin{cases} 0 & x < 0 \\ \frac{\lfloor mx \rfloor + 1}{m+1} & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \\ F_Q(x) &= \sum_{m=1}^{\infty} \frac{\boldsymbol{P}\{M = m\}}{m+1} \sum_{n=0}^{m} \vartheta\left(mx - n\right) \\ &= \begin{cases} 0 & x < 0 \\ \sum_{m \geq 1} \boldsymbol{P}\{M = m\}\, \frac{\lfloor mx \rfloor + 1}{m+1} & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \end{aligned} \tag{13}$$

and the probability (11) with $0 \leq a < b \leq 1$ becomes

$$\boldsymbol{P}\{a < Q \leq b\} = \sum_{m=1}^{\infty} \boldsymbol{P}\{M = m\}\, \frac{\lfloor mb \rfloor - \lfloor ma \rfloor}{m+1} \tag{14}$$

By denoting now as $p_m = \boldsymbol{P}\{M = m\}$ the distribution of M, and as $s = \sup_m p_m$ the supremum of all its values, let us take now a sequence of denominators $\{M_k\}_{k \geq 1}$ with distributions $\{p_m(k)\}_{k \geq 1}$, and with s_k vanishing for $k \to \infty$ in such a way that

$$\lim_{k} s_k \ln k = 0 \tag{15}$$

In other words we consider a sequence of distributions that are increasingly (and uniformly) flattened toward zero, so that the denominators too are increasingly equiprobable. Ready examples of these sequences with $k = 1, 2, \ldots$ are for instance the *geometric* distributions

$$p_m(k) = w_k(1 - w_k)^{m-1} \qquad\qquad m = 1, 2, \ldots$$

with infinitesimal w_k, and the *Poisson* distributions

$$p_m(k) = e^{-\lambda_k} \frac{\lambda_k^{m-1}}{(m-1)!} \qquad\qquad m = 1, 2, \ldots$$

with divergent λ_k

Lemma A.1. *Within the previous notations and conditions we have*

$$\mu_k = \boldsymbol{E}\left[{}^1/_{M_k}\right] = \sum_{m=1}^{\infty} \frac{p_m(k)}{m} \xrightarrow{k} 0 \tag{16}$$

Proof: The positive series defining μ_k is certainly convergent because

$$\mu_k = \sum_{m=1}^{\infty} \frac{p_m(k)}{m} < \sum_{m=1}^{\infty} p_m(k) = 1$$

and hence we can always write

$$\mu_k = \sum_{m=1}^{\infty} \frac{p_m(k)}{m} = \sum_{m=1}^{k} \frac{p_m(k)}{m} + R_k$$

where

$$R_k = \sum_{m=k+1}^{\infty} \frac{p_m(k)}{m} \xrightarrow{k} 0$$

is an infinitesimal remainder. On the other hand, under our stated conditions

$$\sum_{m=1}^{k} \frac{p_m(k)}{m} < s_k \sum_{m=1}^{k} \frac{1}{m} = s_k H_k$$

where H_k denotes the k^{th} *harmonic number*, namely the sum of the reciprocal integers up to ${}^1/_k$: it is well known ([10] **0.131**) that for $k \to \infty$ the H_k grow as $\ln k$, so that from (15) we have $s_k H_k \xrightarrow{k} 0$, and finally $\mu_k = s_k H_k + R_k \xrightarrow{k} 0$ ■

Proposition A.2. *If* $Q = {}^N/_M$ *and* $F_Q(x)$ *is its cdf, then, within the notation and conditions outlined above, we have*

$$\lim_k \boldsymbol{P}\{Q = q\} = 0 \qquad\qquad \lim_k \boldsymbol{P}\{a < Q \le b\} = b - a \tag{17}$$

$$\lim_k F_Q(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x \end{cases} \tag{18}$$

Proof: Since our series have positive terms the first result in (17) follows from (12) and (16) because, with $q \doteq {}^n/_j$

$$\boldsymbol{P}\{Q = q\} = \sum_{\ell=1}^{\infty} \frac{p_{\ell j}(k)}{\ell j + 1} < \sum_{m=1}^{\infty} \frac{p_m(k)}{m+1} < \sum_{m=1}^{\infty} \frac{p_m(k)}{m} = \mu_k \xrightarrow{k} 0$$

As for the second result in (17), since for every real number x it is $x-1 \leq \lfloor x \rfloor \leq x$, for every $k = 1, 2, \ldots$, and $0 \leq a < b \leq 1$, we have from (14)

$$\sum_{m=1}^{\infty} p_m(k)\frac{m(b-a)-1}{m+1} \leq \ \boldsymbol{P}\{a < Q \leq b\} \ \leq \sum_{m=1}^{\infty} p_m(k)\frac{m(b-a)+1}{m+1}$$

namely

$$b-a+(a-b-1)\sum_{m=1}^{\infty}\frac{p_m(k)}{m+1} \leq \ \boldsymbol{P}\{a < Q \leq b\} \ \leq b-a+(a-b+1)\sum_{m=1}^{\infty}\frac{p_m(k)}{m+1}$$

so that, since $a-b-1 \leq 0$ and $a-b+1 \geq 0$, it is

$$b-a+(a-b-1)\mu_k \leq \ \boldsymbol{P}\{a < Q \leq b\} \ \leq b-a+(a-b+1)\mu_k$$

The second result (17) follows then from (16). In a similar way we finally find for (18) that

$$\sum_{m=1}^{\infty} p_m(k)\frac{mx}{m+1} \leq F_Q(x) \leq \sum_{m=1}^{\infty} p_m(k)\frac{mx+1}{m+1} \qquad\qquad 0 \leq x \leq 1$$

namely

$$x - x\sum_{m=1}^{\infty}\frac{p_m(k)}{m+1} \leq F_Q(x) \leq x + (1-x)\sum_{m=1}^{\infty}\frac{p_m(k)}{m+1}$$

and hence

$$x - x\,\mu_k < F_Q(x) < x + (1-x)\mu_k$$

so that the result again follows from (16) ■

From this proposition we see that in the limit $k \to \infty$, while the probability of every single rational number rightly vanishes, the probability of these numbers lumped together in intervals does not: a behavior highly reminiscent of what happens to continuously distributed *real rv*'s. For the reasons presented at the beginning of this appendix, however, the previous result by no means imply that we can implement a uniform limit distribution on $\mathbb{Q}_0$ (as we said: there is not such a thing), but it rather suggests that our random rational numbers Q – at least for denominators m distributed in a fairly flat way, and numerators n conditionally equiprobable between 0 and m – asymptotically behave as uniformly distributed in $[0, 1]$, and hence they quite reasonably correspond to our intuitive idea of *taking rational numbers at random.*

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