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Coupling Poisson Processes by Self-Decomposability

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Abstract. We analyze a method to produce pairs of non-independent Poisson processes $M(t), N(t)$ from positively correlated, self-decomposable, exponential renewals. In particular, the present paper provides the family of copulas pairing the renewals, along with the closed form for the joint distribution $p_{m,n}(s,t)$ of the pair $(M(s), N(t))$, an outcome which turns out to be instrumental to produce explicit algorithms for applications in finance and queuing theory. We finally discuss the cross-correlation properties of the two processes and the relative timing of their jumps.

Mathematics Subject Classification. Primary 60G55, Secondary 60G50, 60J75, 60K05, 91G60.

Keywords. Point processes, Renewal theory, Correlated Poisson processes.

1. Introduction

Recent studies have shown that the spot dynamics of commodity markets displays mean reversion, seasonality and jumps [1], and some methodologies have been proposed to take dependency into account based on correlation and co-integration [2,3]. However, these approaches can become mathematically cumbersome and non-treatable when leaving the Gaussian-Itô world. In this context, it has been indeed recently proposed [4] to consider 2-dimensional jump diffusion processes with a 2-dimensional Gaussian and a 2-dimensional compound Poisson component, and, as also suggested in different circumstances [5], we show here that a revealing approach to model the dependency of the 2-dimensional Poisson processes can be supplied on the ground of the *self-decomposability* of the exponential random variables used for its construction.

The present paper is in particular devoted to find both the copula function pairing our correlated renewals, and an explicit form for the joint distribution $p_{m,n}(s,t)$ of our pair of correlated Poisson processes $M(s), N(t)$: this will prove to be instrumental to produce the efficient algorithms that can be

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used in financial applications [4]. If indeed the pairs of correlated, exponential random variables (*rv*'s) (X_k, Y_k)—used to produce the renewals in our processes—are interpreted as random waiting times with *random delays*; the proposed model can help describing their co-movement and can answer some common questions arising in the financial context:

- Once a financial institution defaults how long should one wait for a dependent institution to default too?
- A market receives a news interpreted as a shock: how long should one wait to see the propagation of that shock onto a dependent market?
- What is the impact of the correlations among the shocks for different insurance companies on a fair assessment of the risk of losses [6]?

It is worth noticing, moreover, that we achieve our aim of producing a 2-dimensional Poisson process with *dependent* marginals without resorting to an a priori copula (distributional) approach: the dependence among arrival times will indeed be made explicit in terms of combinations of *rv*'s, and we only recover and discuss the corresponding copula functions as an outcome of this model. As a consequence, because of this $\boldsymbol{P}$-a.s. relationship between the random times, the two Poisson processes can be seen as linked with a form of *co-integration* between their jumps. Similar models—albeit rather less sophisticated—were also used in order to model a multi-component reliability system [5], while the so-called *Common Poisson Shock Models* [7] are in fact quite different from that presented here.

The main practical consequence of our results is then that the price and the Greeks of the spread options considered in the applications [4] can be calculated in *closed form* using either the Margrabe formula (if the strike is zero), or some well-known approximation [8]. In any case, our model entails explicit algorithms for the simulation of correlated Poisson processes, and can be used in the *Monte Carlo* simulations. An extension to the multi-dimensional case, as well as to different dynamics other than Poisson, will be considered in future studies, but, under the assumption that only two underlyings have jump component, the price and the Greeks of spread options can be obtained even now by the moment-matching methodology recently proposed in [9].

The paper is organized as follows: in Sect. 2, we first show how (hitherto *positively*) correlated exponential *rv*'s can be deduced from the self-decomposability of their laws; then in Sect. 3, we briefly discuss the copula functions produced by this model. Using pairs of these exponential *rv*'s as correlated renewals, in Sect. 4 we subsequently produce a 2-dimensional Poisson process with correlated components, and in Sect. 5 we explicitly deduce their joint distributions. Finally in Sect. 6, the cross-correlation properties of the Poisson processes are briefly analyzed, and the relative timing of their jumps is used to shed new light on the dependence mechanism of a model allowing for the possibility of a delayed propagation of correlated shocks. We conclude by pointing out first that we would also be able to produce other correlated *rv*'s (Erlang, Gamma, $EPT\dots$) by making use, once more, of their self-decomposability, and then that the results of this paper should also be extended to *negatively* correlated renewals, a possibility—not open to other procedures—that will be postponed to future inquiries. Lengthy proofs are confined in the Appendices, together with a few details about the notation adopted throughout the paper.

2. Correlation from Self-Decomposability

2.1. Joint Distributions

A law with density (*pdf*) $f(x)$ and characteristic function (*chf*) $\varphi(u)$ is said to be *self-decomposable* (*sd*) [10,11] when for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u)$$

This is a well-known family of laws with many relevant properties. We will also say that a random variable (*rv*) X with *pdf* $f(x)$ and *chf* $\varphi(u)$ is *sd* when its law is *sd*: looking at the definition this means that for every $0 < a < 1$ we can always find two *independent rv*'s Y (with the same law of X) and Z_a (with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$) such that

$$X \stackrel{d}{=} aY + Z_a$$

We can look at this, however, also from a different perspective: if Y is *sd* and, to the extent that, for $0 < a < 1$, an independent Z_a with the suitable law is known, we can define a third *rv*

$$X \equiv aY + Z_a \qquad \boldsymbol{P}\text{-a.s.}$$

being sure that it will have the same law as Y. In the following, we will mainly adopt this second standpoint.

We turn now, for later convenience, to give the joint laws of the triplet (X, Y, Z_a): for the *chf* $\psi(u, v, w)$ we easily find from the independence of Y and Z_a that

$$\begin{aligned}\psi(u, v, w) &= \boldsymbol{E}\left[e^{i(uX+vY+wZ_a)}\right]\\ &= \varphi(au + v)\chi_a(u + w) = \varphi(au + v)\frac{\varphi(u + w)}{\varphi(a(u + w))}\end{aligned}$$

while the marginal, joint *chf*'s of the pairs (X, Y) and (X, Z_a), respectively, are

$$\phi(u, v) = \psi(u, v, 0) = \varphi(au + v)\frac{\varphi(u)}{\varphi(au)}$$

$$\omega(u, w) = \psi(u, 0, w) = \varphi(au)\frac{\varphi(u + w)}{\varphi(a(u + w))}$$

As for the *pdf* $\kappa(x, y, z)$, on the other hand, we have from an inverse Fourier transform ($\delta(x)$ is the *Dirac delta* distribution; here and in the following the computational details can be retrieved from an extended preprint version [12])

$$\kappa(x, y, z) = f(y)\, g_a(x - ay)\, \delta[z - (x - ay)]$$

so that the marginal, joint *pdf*'s of (X, Y) and (X, Z_a) will, respectively, be

$$h(x,y) = f(y)\, g_a(x - ay) \qquad\qquad \ell(x,z) = \frac{1}{a}\, f\left(\frac{x-z}{a}\right) g_a(z) \tag{2.1}$$

Finally, the joint cumulative distribution function (*cdf*) of (X, Y) is

$$H(x,y) = \int_{-\infty}^{y} f(y')\, G_a(x - ay')\, \mathrm{d}y' \qquad\qquad G_a(z) = \int_{-\infty}^{z} g_a(z')\, \mathrm{d}z'$$

where $G_a(z)$ is the *cdf* of Z_a. The particular form of $H(x, y)$ will be instrumental in finding the copula functions [13] eventually pairing X and Y.

We can finally also calculate the correlation coefficients r_{XY} and r_{XZ_a}: if we put $\boldsymbol{E}[X] = \boldsymbol{E}[Y] = \mu$ and $\boldsymbol{V}[X] = \boldsymbol{V}[Y] = \sigma^2$, from the Y, Z_a independence we have

$$\boldsymbol{E}[XY] = \boldsymbol{E}[(aY + Z_a)Y] = a\sigma^2 + \mu^2$$

and hence $r_{XY} = a$. In a similar vein, to calculate r_{XZ_a} we first remark that $\boldsymbol{V}[X] = a^2\boldsymbol{V}[Y] + \boldsymbol{V}[Z_a]$, namely $\boldsymbol{V}[Z_a] = (1 - a^2)\sigma^2$, and then from

$$\boldsymbol{E}[XZ_a] = \boldsymbol{E}[(aY + Z_a)Z_a] = (1 - a^2)\sigma^2 + (1 - a)\mu^2$$

we finally find $r_{XZ_a} = 1 - a^2$.

2.2. An Example: The Exponential Laws $\mathfrak{E}_1(\lambda)$

It is well known that the exponential laws $\mathfrak{E}_1(\lambda)$ with *pdf* and *chf* (see Appendix A for the notations adopted from now on)

$$\lambda f_1(\lambda x) = \lambda e^{-\lambda x} \vartheta(x) \qquad\qquad \varphi_1(u/\lambda) = \frac{\lambda}{\lambda - iu}$$

are a typical example of *sd* laws [10], and in this case we can explicitly give the law of Z_a: we have indeed

$$\chi_a(u) = \frac{\varphi_1(u/\lambda)}{\varphi_1(au/\lambda)} = \frac{\lambda - iau}{\lambda - iu} = a + (1-a)\frac{\lambda}{\lambda - iu} = a + (1-a)\varphi_1(u/\lambda) \tag{2.2}$$

which (for $0 < a < 1$) is a mixture of a law $\boldsymbol{\delta}_0$ degenerate in 0, and an exponential $\mathfrak{E}_1(\lambda)$, namely

$$Z_a \sim a\boldsymbol{\delta}_0 + (1-a)\mathfrak{E}_1(\lambda)$$

so that its *pdf* and *cdf* respectively, are

$$g_a(z) = a\delta(z) + (1-a)\lambda e^{-\lambda z}\vartheta(z)$$
$$G_a(z) = \left[a + (1-a)(1 - e^{-\lambda z})\right]\vartheta(z)$$

It is also easy to prove, on the other hand, that this coincides with the law of the product of two other independent *rv*'s: an exponential $Z \sim \mathfrak{E}_1(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$ with $a = \boldsymbol{P}\{B(1) = 0\}$, so that we can always write

$$Z_a = B(1)\, Z$$

In short, given two exponential *rv*'s $Y \sim \mathfrak{E}_1(\lambda)$ and $Z \sim \mathfrak{E}_1(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$, all three mutually independent, the *rv* X defined as

$$X \equiv aY + B(1)Z \tag{2.3}$$

is again an exponential $\mathfrak{E}_1(\lambda)$. From (2.1), we also find that the joint *pdf* of X, Y is

$$h(x,y) = \lambda e^{-\lambda y}\vartheta(y)\left[a\delta(x-ay) + (1-a)\lambda e^{-\lambda(x-ay)}\vartheta(x-ay)\right]$$

and hence its joint *cdf* is

$$\begin{aligned} H(x,y) &= \int_{-\infty}^{y} \lambda e^{-\lambda y'}\vartheta(y')\left[a + (1-a)(1-e^{-\lambda(x-ay')})\right]\vartheta(x-ay')\,\mathrm{d}y' \\ &= \vartheta\left(y \wedge \frac{x}{a}\right)\left[\left(1-e^{-\lambda\left(y\wedge\frac{x}{a}\right)}\right) - e^{-\lambda x}\left(1-e^{-\lambda(1-a)\left(y\wedge\frac{x}{a}\right)}\right)\right] \end{aligned}$$

Of course this is far from the only possible joint law with exponential marginals (see also Sect. 3.1), but it is noticeable because it traces its origins back to a model of self-decomposability of the exponentials. As for the correlations among X, Y and Z, we already know that $r_{XY} = a$. For r_{XZ}, we first find that

$$\boldsymbol{E}\left[XZ\right] = \boldsymbol{E}\left[(aY + Z_a)Z\right] = \frac{2-a}{\lambda^2}$$

and then that

$$r_{XZ} = 1 - a = 1 - r_{XY}$$

so that for our three exponentials in (2.3), we eventually have

$$r_{XY} + r_{XZ} = 1 \qquad r_{XY} = a \qquad r_{YZ} = 0.$$

2.3. Positively Correlated Exponential rv's

It is apparent now from the discussion in the previous section that the self-decomposability of the exponential laws $\mathfrak{E}_1(\lambda)$ can be turned into a simple procedure to generate identically distributed and correlated *rv*'s: given $Y \sim \mathfrak{E}_1(\lambda)$, in order to produce another $X \sim \mathfrak{E}_1(\lambda)$ with correlation $0 < a < 1$, it would be enough to take $Z \sim \mathfrak{E}_1(\lambda)$ and $B(1) \sim \mathfrak{B}(1, 1-a)$ independent from Y and define X as in (2.3). In other words, X will be nothing else than the exponential Y *down a-rescaled*, plus another independent exponential Z *randomly intermittent with frequency* $1-a$. The self-decomposability of the exponential laws ensures then that, for every $0 < a < 1$, also X marginally is an $\mathfrak{E}_1(\lambda)$. Remark that we would not have the same result by taking more naive combinations of Y and Z. Consider for instance the sum $aY + (1-a)Z$ of our two independent, exponential *rv*'s: in this case, since $aY \sim \mathfrak{E}_1\left(\frac{\lambda}{a}\right)$ and $(1-a)Z \sim \mathfrak{E}_1\left(\frac{\lambda}{1-a}\right)$, the law of $aY + (1-a)Z$ would be $\mathfrak{E}_1\left(\frac{\lambda}{a}\right) * \mathfrak{E}_1\left(\frac{\lambda}{1-a}\right)$, which is neither an exponential $\mathfrak{E}_1(\lambda)$, nor even in general an Erlang $\mathfrak{E}_2$ because of the difference between the two parameters.

The proposed procedure can now be adapted to generate a sequence of independent pairs of exponential *rv*'s, with correlated components, (X_k, Y_k),

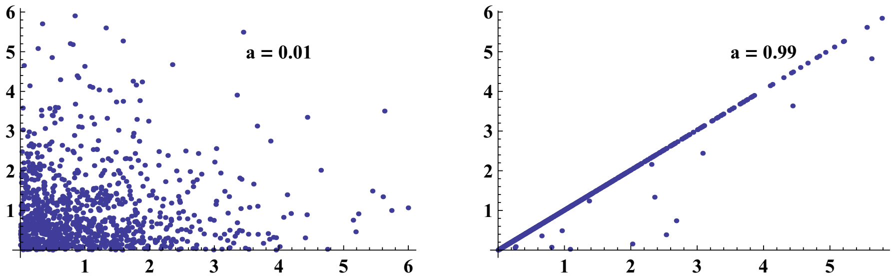

Figure 1. The pairs (X_k, Y_k) with correlation 0.01 and 0.99

$k = 1, 2, \dots$ (or, if we prefer, X_k, Z_k) that will act in the subsequent sections as renewals for a two-dimensional point (Poisson) process: take indeed $0 < a < 1$, produce two independent *id* exponentials Y, Z and another independent Bernoulli $B(1)$, then define $X = aY + B(1)Z$ and take the pair X, Y. By independently replicating this procedure, we will get a sequence of *iid* two-dimensional pairs (X_k, Y_k) that will be used later to generate a two-dimensional Poisson process with correlated renewals. Of course—not surprisingly—the case of uncorrelated pairs of renewals (X_k, Y_k), and hence of independent Poisson processes, is retrieved from our model in the limit $a \to 0$, because in this case we just have $X = Z$ which is by definition independent from Y. On the other hand, it is also apparent that in the opposite limit $a \to 1$ (namely when r_{XY} goes to 1) we tend to have $X = Y$, $\boldsymbol{P}$-a.s. so that the time pairs will fall precisely on the diagonal of the two-dimensional time, and the two Poisson processes will simply $\boldsymbol{P}$-a.s. coincide. To see it from another standpoint, we could look to some simulation of the pairs (X_k, Y_k): for small correlations a, the scatter plot of our pairs (X_k, Y_k) tends to evenly spread out within the first quadrant without any apparent hint to some form of dependence; on the other hand for a near to 1 the points tend to cluster together along the diagonal, as can be seen in the Fig. 1.

3. Copulas for Bivariate Exponentials

3.1. A Family of Copula Functions

From the discussion in Sect. 2.2, we know that the pair X, Y of correlated *rv*'s deduced from their self-decomposability has the joint *cdf*

$$H(x, y) = \vartheta\left(y \wedge \frac{x}{a}\right)\left[\left(1 - e^{-\lambda\left(y \wedge \frac{x}{a}\right)}\right) - e^{-\lambda x}\left(1 - e^{-\lambda(1-a)\left(y \wedge \frac{x}{a}\right)}\right)\right] \quad (3.1)$$

with the exponential marginal *cdf*'s (the notation is here slightly simplified)

$$F(x) = \vartheta(x)\left(1 - e^{-\lambda x}\right) \qquad G(y) = \vartheta(y)\left(1 - e^{-\lambda y}\right) \quad (3.2)$$

To find out the *copula function* $C(u, v)$ [13] pairing X, Y, we then first remark that

$$e^{-\lambda x} = 1 - F(x) \qquad e^{-\lambda y} = 1 - G(y) \qquad e^{-a\lambda y} = [1 - G(y)]^a$$

while

$$
\begin{aligned}
0 \le \frac{x}{a} \le y \quad &\Longleftrightarrow \quad e^{-a\lambda y} \le e^{-\lambda x} \quad &\Longleftrightarrow \quad [1-G(y)]^a \le 1-F(x)\\
0 \le y \le \frac{x}{a} \quad &\Longleftrightarrow \quad e^{-a\lambda y} \ge e^{-\lambda x} \quad &\Longleftrightarrow \quad [1-G(y)]^a \ge 1-F(x)
\end{aligned}
$$

and then that our joint *cdf* (3.1) takes the form

$$
\begin{aligned}
H(x,y) &= F(x) &\text{for } [1-G(y)]^a \le 1-F(x)\\
H(x,y) &= F(x) - [1-G(y)]\left(1 - \frac{1-F(x)}{[1-G(y)]^a}\right) &\text{for } [1-G(y)]^a \ge 1-F(x)
\end{aligned}
$$

which can also be conveniently summarized as:

$$
H(x,y) = F(x) - [1-G(y)]\left(1 - \frac{1-F(x)}{[1-G(y)]^a}\right)^+
$$

As a consequence, we get the following family of copula functions:

$$
C_a(u,v) = u - (1-v)\left[1 - \frac{1-u}{(1-v)^a}\right]^+ = u - \frac{[(1-v)^a - (1-u)]^+}{(1-v)^{a-1}} \tag{3.3}
$$

which for $0 \le a \le 1$ runs between two extremal copulas

$$
\begin{aligned}
C_0(u,v) &= uv &&\text{independent marginals}\\
C_1(u,v) &= u \wedge v &&\text{fully positively correlated marginals}
\end{aligned}
$$

It is easy to see that $C_1(u,v)$ also coincides with the Fréchet–Höffding upper bound $\overline{C}(u,v)$ for copulas (see Sect. 3.3).

3.2. Bivariate Exponential Distributions

Several examples—all different from (3.3)—of bivariate distributions with exponential marginals $\mathfrak{E}_1(\lambda)$ and $\mathfrak{E}_1(\mu)$ can be found in the literature [13,14]. First, we find the *Gumbel bivariate exponential distribution* [13,15] with $0 \le a \le 1$ and

$$
\begin{aligned}
H(x,y) &= \vartheta(x)\vartheta(y)\left(1 - e^{-\lambda x} - e^{-\mu y} + e^{-\lambda x - \mu y - a\lambda\mu\, xy}\right)\\
h(x,y) &= \vartheta(x)\vartheta(y)\left[a(\lambda x + \mu y + a\lambda\mu\, xy) + 1 - a\right] e^{-\lambda x - \mu y - a\lambda\mu\, xy}\\
C_a(u,v) &= u + v - 1 + (1-u)(1-v)e^{-\frac{a}{\lambda\mu}\ln(1-u)\ln(1-v)}
\end{aligned}
$$

It is apparent that $C_0(u,v) = uv$ gives the independent exponentials, while

$$
C_1(u,v) = u + v - 1 + (1-u)(1-v)e^{-\frac{1}{\lambda\mu}\ln(1-u)\ln(1-v)}
$$

does not seem to correspond to some notable copula. Then, there is the *Marshall–Olkin bivariate exponential distribution* [13,16,17] with $0 \le a, b \le 1$ and

$$
H(x,y) = \begin{cases} \vartheta(x)\vartheta(y)(1-e^{-\lambda x})^{1-a}(1-e^{-\mu y}) & \text{if } (1-e^{-\lambda x})^a \ge (1-e^{-\mu y})^b\\ \vartheta(x)\vartheta(y)(1-e^{-\lambda x})(1-e^{-\mu y})^{1-b} & \text{if } (1-e^{-\lambda x})^a \le (1-e^{-\mu y})^b \end{cases}
$$

$$
h(x,y) = \begin{cases} \frac{1-a}{(1-e^{-\lambda x})^a}\,\vartheta(x)\lambda e^{-\lambda x}\,\vartheta(y)\mu e^{-\mu y} & \text{if } (1-e^{-\lambda x})^a \ge (1-e^{-\mu y})^b\\ \frac{1-b}{(1-e^{-\mu y})^b}\,\vartheta(x)\lambda e^{-\lambda x}\,\vartheta(y)\mu e^{-\mu y} & \text{if } (1-e^{-\lambda x})^a \le (1-e^{-\mu y})^b \end{cases}
$$

$$
C_{a,b}(u,v) = (u^{1-a}v) \wedge (uv^{1-b}) \begin{cases} u^{1-a}v & \text{when } u^a \ge v^b\\ uv^{1-b} & \text{when } u^a \le v^b \end{cases}
$$

In this case, $C_{0,0}(u,v) = uv$ again is the independent copula, while $C_{1,1}(u,v) = u \wedge v$ is the Fréchet–Höffding upper bound $\overline{C}(u,v)$ (see Sect. 3.3): apart from these extremal values, however, also this Marshall–Olkin copula differs from (3.3). A third family of copulas can finally be traced back to the *Raftery bivariate exponential distribution* [13,18,19]: in this case, the copula functions are

$$C_a(u,v) = u \wedge v + \frac{a}{2-a}(uv)^{\frac{1}{a}} \left[1 - (u \vee v)^{1-\frac{2}{a}}\right]$$

and correspond to the case of correlated exponential *rv*'s X, Y which are produced by three independent exponential *rv*'s U, V and Z according to the definitions

$$X \equiv aU + B(1)Z \qquad\qquad Y \equiv aV + B(1)Z$$

Here, at variance with our model based on self-decomposability, the correlation is apparently produced by the presence of the same exponential *rv* Z in both the right-hand sides of the definitions. In short, it results from these examples that our family of copulas (3.3) seems not to have been used in advance to couple pairs of marginal exponentials.

3.3. Fréchet–Höffding Bounds

It is well known [13] that every copula function $C(u,v)$ falls between the Fréchet–Höffding bounds

$$\underline{C}(u,v) = (u+v-1)^+ \leq C(u,v) \leq u \wedge v = \overline{C}(u,v)$$

and we have also found in the Sect. 3.1 that the copula $C_1(u,v)$ for our fully correlated ($r_{XY} = 1$) exponential marginals coincides with the Fréchet–Höffding upper bound. By keeping in mind a possible generalization of our model to the case of *negatively correlated* exponentials, we will briefly recall in this section a few general features of the joint *cdf*'s $H(x,y) = C(F(x), G(y))$ produced by the pairing of two given *cdf*'s $F(x)$ and $G(x)$ by means of the Fréchet–Höffding lower and upper bounds.

Let us suppose for simplicity that $F(x)$ and $G(x)$ are continuous and strictly increasing functions so that the inverse functions exist, and consider first the lower bound copula $\underline{C}(u,v) = (u+v-1)^+$: in that case, the condition $F(x) + G(y) \geq 1$ is equivalent to both the inequalities

$$x \geq \beta(y) = F^{-1}(1 - G(y)) \qquad\qquad y \geq \alpha(x) = G^{-1}(1 - F(x))$$

and hence from $H(x,y) = (F(x) + G(y) - 1)^+$ we first have

$$\partial_x H(x,y) = \begin{cases} f(x) & \text{if } \ y \geq \alpha(x) \\ 0 & \text{if } \ y < \alpha(x) \end{cases}$$

$$\partial_y H(x,y) = \begin{cases} g(y) & \text{if } \ x \geq \beta(y) \\ 0 & \text{if } \ x < \beta(y) \end{cases}$$

where $f(x)$ and $g(y)$ are the corresponding marginal *pdf*'s, and then the joint *pdf* is

$$h(x,y) = \partial_x \partial_y H(x,y) = f(x)\delta(y - \alpha(x)) = g(y)\delta(x - \beta(y)) \qquad (3.4)$$

As a consequence we can say that the joint laws produced by the copula $\underline{C}(u,v)$ describe pairs of coupled *rv*'s X, Y satisfying $\boldsymbol{P}$-a.s. the functional relations

$$X=\beta(Y)=F^{-1}(1-G(Y)) \qquad\qquad Y=\alpha(X)=G^{-1}(1-F(X)). \quad (3.5)$$

A formally identical result can be proved for the case of the upper bound 2-copula $\overline{C}(u,v)=u\wedge v$ but for the fact that now the functions $\alpha(x)$ and $\beta(y)$ must be redefined as:

$$\alpha(x)=G^{-1}(F(x)) \qquad\qquad \beta(y)=F^{-1}(G(y)).$$

In the case of the lower bound copula $\underline{C}(u,v)$, it is interesting to remark now that when for instance $F(x)$ and $G(y)$ are Gaussian *cdf*'s the functions $\alpha(x)$ and $\beta(y)$ are linear with negative proportionality coefficients, so that the pair X, Y is perfectly anti-correlated with $r_{XY}=-1$. The same happens also in the case of a pair of Student laws of the same order. This is true indeed for every other pair of marginal laws of the *same type* and with *support coincident with* $\boldsymbol{R}$. On the other hand, when the marginals either are not of the same type, or have an unbounded support strictly contained in $\boldsymbol{R}$ (as happens for exponential laws), they apparently cannot reciprocally be in a linear relation with negative proportionality coefficient and, hence, cannot be totally *linearly* anti-correlated. In this case, it can still be proved by means of Höffding's Lemma (see [13, p. 190]) that the minimal correlation is reached by means of the lower bound copula $\underline{C}$, but now $\alpha(x)$ and $\beta(y)$ can no longer be *linear* functions, and r_{XY} will be strictly larger than -1. By taking indeed the Fréchet–Höffding lower bound $\underline{C}(u,v)=(u+v-1)^+$ as the copula for our exponentials (3.2), we would find the *pdf* (3.4) and the functional relations (3.5) where now

$$\alpha(x)=-\frac{1}{\lambda}\ln\left(1-e^{-\lambda x}\right) \qquad\qquad \beta(y)=-\frac{1}{\lambda}\ln\left(1-e^{-\lambda y}\right)$$

and a short calculation would then show that in this case

$$r_{XY}=1-\frac{\pi^2}{6}\approx-0.645$$

so that this minimal anti-correlation allowed for exponential *rv*'s would in any case be larger than -1. It could in fact be proved in general (see [13, p. 30–32]) that, when X and Y are continuous, Y is almost surely a decreasing function of X if and only if the copula of X and Y is $\underline{C}$. Random variables with copula $\underline{C}$ are often called countermonotonic. We postpone to a subsequent enquiry a detailed study of negatively correlated exponentials.

4. Correlated Poisson Processes

Following the discussion of Sect. 2, it is easy now to produce a sequence of *rv*'s by independently iterating the definition (2.3)

$$X_k=aY_k+B_k(1)Z_k \qquad\qquad k=1,2,\ldots \quad (4.1)$$

in such a way that for every k: X_k, Y_k, Z_k are $\mathfrak{E}_1(\lambda)$, $B_k(1)$ are $\mathfrak{B}(1,1-a)$, and $Y_k, Z_k, B_k(1)$ are mutually independent. The pairs (X_k, Y_k) instead will

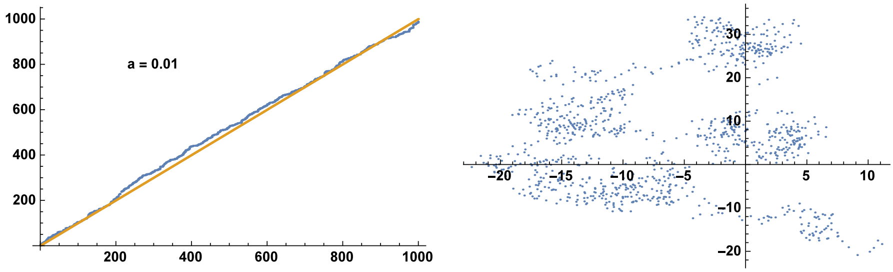

Figure 2. Sample pairs of the two-dimensional point process (T_n, S_n) with correlation $r_{XY} = 0.01$: on the *left*, the points are compared with the average trend $(\frac{n}{\lambda}, \frac{n}{\mu})$; on the *right*, they are instead plotted after centering around these averages

be a-correlated for every k. Add moreover $X_0 = Y_0 = Z_0 = 0$, $\boldsymbol{P}$-a.s. to the list, and take then the point processes for $n = 0, 1, 2, \dots$

$$T_n = \sum_{k=0}^{n} X_k \qquad S_n = \frac{\lambda}{\mu}\sum_{k=0}^{n} Y_k \qquad R_n = \sum_{k=0}^{n} Z_k. \tag{4.2}$$

Since the $X_k \sim \mathfrak{E}_1(\lambda)$ are *iid rv*'s, we know that $T_n \sim \mathfrak{E}_n(\lambda)$ are distributed as Erlang (gamma) laws with *pdf*'s $\lambda f_n(\lambda x)$ and *chf*'s $\varphi_n(u/\lambda)$ (see Appendix A for notations) where it is understood that $T_0 \sim \mathfrak{E}_0 = \boldsymbol{\delta}_0$. In a similar way, we can argue that $S_n \sim \mathfrak{E}_n(\mu)$ and $R_n \sim \mathfrak{E}_n(\lambda)$. We will finally denote with $N(t) \sim \mathfrak{P}(\lambda t)$ and $M(t) \sim \mathfrak{P}(\mu t)$ the *correlated Poisson processes* associated, respectively, with T_n and S_n.

In order to get a first look to these processes, we generate $n = 1\,000$ pairs (X_k, Y_k) with the associated two-dimensional point process (T_n, S_n), and then we simulate the corresponding Poisson processes $N(t)$ and $M(t)$. The pairs (T_n, S_n) are first plotted along with their average time increases $(\frac{n}{\lambda}, \frac{n}{\mu})$, and then after centering around these averages, namely as

$$T_n - \frac{n}{\lambda} \qquad S_n - \frac{n}{\mu} \qquad n = 1, 2, \dots, 1\,000$$

In this second rendering, the random behavior is magnified by consistently reducing the plot scale to a suitable size. In the same way for the Poisson processes, we first show samples of the pair $N(t), M(t)$, and then that of their *compensated* versions $\widetilde{N}(t) = N(t) - \lambda t$ and $\widetilde{M}(t) = M(t) - \mu t$.

In Fig. 2, we plotted the two-dimensional point process (T_n, S_n) with $\lambda = \mu = 1$ and $a = r_{XY} = 0.01$: since the correlation among the renewals is negligible, the right-hand plots (centered around the averages) apparently show a random behavior. In Fig. 3, instead we took $a = r_{XY} = 0.99$, namely we generated strongly and positively correlated renewals. In this second case, as it was to be expected, the centered time pairs fall into line among themselves. As for the Poisson processes themselves, in Fig. 4, the trajectories on the left-hand side have $a = 0.01$ correlation and look fairly independent, after

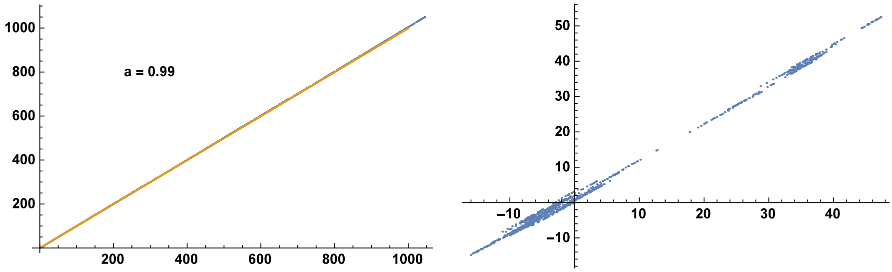

Figure 3. Sample pairs of the two-dimensional point process (T_n, S_n) with correlation $r_{XY} = 0.99$: on the *left*, the points are compared with the average trend $(\frac{n}{\lambda}, \frac{n}{\mu})$; on the *right*, they are instead plotted after centering around these averages

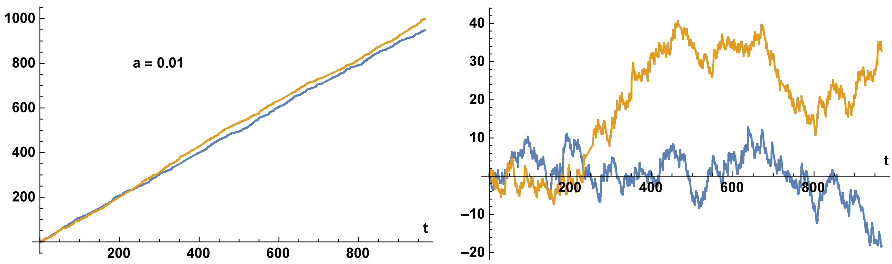

Figure 4. On the *left*, sample paths of the two Poisson processes $N(t)$ and $M(t)$ with correlation $r_{XY} = 0.01$ are shown; on the *right*, we instead have the corresponding compensated Poisson processes $\widetilde{N}(t)$ and $\widetilde{M}(t)$

suitable compensation, of the right-hand side. In Fig. 5, instead we took a correlation $a = 0.99$ and the compensated trajectories are now almost superimposed. Remark as on the left-hand sides of these figures both the Poisson processes and the time pairs appear to be quite near to one another, and to their averages because of a scale effect which is eliminated by compensation and centering in the corresponding right-hand sides.

Proposition 4.1. *The rv's*

$$\zeta_n \equiv \sum_{k=0}^{n} B_k(1) Z_k$$

turn out to be the sum of a (random) binomial number $B(n) \sim \mathfrak{B}(n, 1-a)$ of iid exponentials $\mathfrak{E}_1(\lambda)$, and hence they follow an Erlang law with a random index $B(n)$ (here $B(0) = 0$), namely

$$\zeta_n = \sum_{k=0}^{B(n)} Z_k \sim \mathfrak{E}_{B(n)}(\lambda)$$

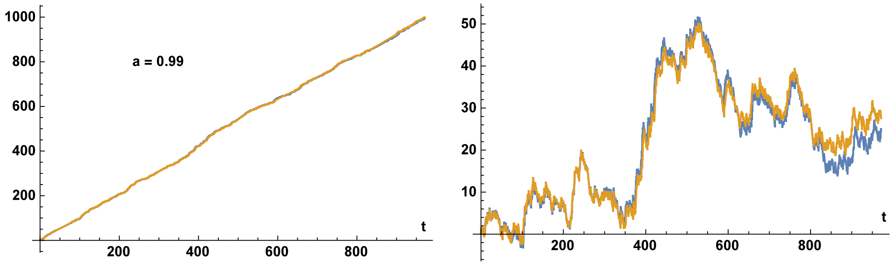

Figure 5. On the *left*, sample paths of the two Poisson processes $N(t)$ and $M(t)$ with correlation $r_{XY} = 0.99$ are shown; on the *right*, we instead have the corresponding compensated Poisson processes $\widetilde{N}(t)$ and $\widetilde{M}(t)$

Proof. This is better seen from the point of view of the mixtures by remarking that, if $\varphi_1(u/\lambda)$ is the *chf* of $\mathfrak{E}_1(\lambda)$, we have from (2.2) (see also Appendix A)

$$\begin{aligned}\varphi_{\zeta_n}(u) &= \boldsymbol{E}\left[e^{iu\zeta_n}\right] = \boldsymbol{E}\left[\prod_{k=0}^{n} e^{iuB_k(1)Z_k}\right] = \prod_{k=0}^{n} \boldsymbol{E}\left[e^{iuB_k(1)Z_k}\right] \\ &= \left[a + (1-a)\varphi_1\left(\frac{u}{\lambda}\right)\right]^n = \sum_{k=0}^{n}\binom{n}{k} a^{n-k}(1-a)^k\varphi_1^k\left(\frac{u}{\lambda}\right) \\ &= \sum_{k=0}^{n}\beta_k(n)\varphi_k\left(\frac{u}{\lambda}\right)\end{aligned}$$

which, if $\varphi_k(u) = [\varphi_1(u)]^k$ are the *chf* of $\mathfrak{E}_k(1)$, eventually is a mixture of Erlang laws $\mathfrak{E}_k(\lambda)$ with the binomial weights $\beta_k(n)$. It is understood here that $\varphi_1^0(u) = 1$, so that $\mathfrak{E}_0(\lambda) = \boldsymbol{\delta}_0$ and $f_0(x) = \delta(x)$ (see Appendix A). □

A straightforward consequence of the previous proposition (which apparently just amounts to acknowledge a subordination) is that now from

$$\sum_{k=0}^{n} X_k = a\sum_{k=0}^{n} Y_k + \sum_{k=0}^{n} B_k(1)Z_k$$

we will also have

$$T_n = \frac{a\mu}{\lambda}S_n + \zeta_n = \frac{a\mu}{\lambda}S_n + \sum_{k=0}^{B(n)} Z_k = \frac{a\mu}{\lambda}S_n + R_{B(n)} \tag{4.3}$$

where $R_{B(n)} \sim \mathfrak{E}_{B(n)}(\lambda)$ is the point process R_n with a random index $B(n)$.

It is worthwhile to notice that the previous results also substantiate the well-known fact that the Erlang *rv*'s are self-decomposable too: the explicit knowledge of the ζ_n law allows indeed to construct pairs of dependent Erlang *rv*'s with correlation a.

5. The Joint Distribution

Our main task is now to explicitly calculate the joint distribution of our Poisson processes at arbitrary times $s,t \geq 0$ and $n,m = 0,1,2,\dots$

$$\begin{aligned} p_{m,n}(s,t) &= \boldsymbol{P}\left\{M(s)=m,\, N(t)=n\right\} \\ &= \boldsymbol{P}\left\{S_m \leq s < S_{m+1},\, T_n \leq t < T_{n+1}\right\} \end{aligned}$$

and to this effect we first remark (in a slightly simplified notation) that

$$\begin{aligned} p_{m,n} &= \boldsymbol{P}\left\{M(s)\geq m,\, N(t)\geq n\right\} - \boldsymbol{P}\left\{M(s)\geq m+1,\, N(t)\geq n\right\} \\ &\quad -\boldsymbol{P}\left\{M(s)\geq m,\, N(t)\geq n+1\right\} + \boldsymbol{P}\left\{M(s)\geq m+1,\, N(t)\geq n+1\right\} \\ &= q_{m,n} - q_{m+1,n} - q_{m,n+1} + q_{m+1,n+1} \end{aligned} \tag{5.1}$$

where

$$q_{m,n}(s,t) = \boldsymbol{P}\left\{M(s)\geq m,\, N(t)\geq n\right\} = \boldsymbol{P}\left\{S_m \leq s,\, T_n \leq t\right\}$$

so that by taking

$$w = \frac{\lambda r}{a} \qquad y = \frac{\lambda t}{a} \qquad z = \frac{\lambda t - a\mu s}{a} < y$$

from (4.3) we are reduced to calculate (see also Appendix A).

$$\begin{aligned} q_{m,n} &= \boldsymbol{P}\left\{S_m \leq s,\, \frac{a\mu}{\lambda}S_n + R_{B(n)} \leq t\right\} \\ &= \sum_{\ell=0}^{n} \beta_\ell(n) \int_0^\infty dr \lambda f_\ell(\lambda r) \end{aligned}$$

$$\begin{aligned} &\boldsymbol{P}\left\{S_m \leq s,\, \frac{a\mu}{\lambda}S_n + R_{B(n)} \leq t \,\middle|\, R_\ell = r, B(n)=\ell\right\} \\ &= \lambda \sum_{\ell=0}^{n} \beta_\ell(n) \int_0^t dr\, f_\ell(\lambda r) \boldsymbol{P}\left\{S_m \leq s,\, S_n \leq \lambda\frac{t-r}{a\mu}\right\} \\ &= a\int_0^y \mathrm{d}w\, h_n(aw) \boldsymbol{P}\left\{S_m \leq \frac{y-z}{\mu},\, S_n \leq \frac{y-w}{\mu}\right\} \end{aligned} \tag{5.2}$$

where $\lambda h_n(\lambda x)$ is the (Erlang binomial mixture) *pdf* of $R_{B(n)}$.

Proposition 5.1. *For $n,m = 0,1,2,\dots$ and $\rho,\tau \geq 0$, we have*

$$\begin{aligned} \boldsymbol{P}\left\{S_m \leq \rho,\, S_n \leq \tau\right\} &= \Pi_{m\vee n}\left(\mu(\rho\wedge\tau)\right) \\ &\quad + \left[\Theta_{n-m}\vartheta(\tau-\rho) + \Theta_{m-n}\vartheta(\rho-\tau)\right] \\ &\quad \times \sum_{k=m\wedge n}^{(m\vee n)-1} \Pi_{(m\vee n)-k}\left(\mu|\rho-\tau|\right)\pi_k\left(\mu(\rho\wedge\tau)\right) \end{aligned}$$

with the notations adopted in the Appendix A *for the Poisson laws.*

Proof. See Appendix B for a detailed proof. □

Of course in (5.2), we take in particular

$$\rho = \frac{y-z}{\mu} = s \qquad \tau = \frac{y-w}{\mu} = \lambda\frac{t-r}{a\mu}$$

It is apparent that this result will be instrumental to calculate first $q_{m,n}(s,t)$ in (5.2), and then the distributions $p_{m,n}(s,t)$.

Proposition 5.2. *If* $a\mu s \geq \lambda t$, *then* $p_{m,n}(s,t) = 0$ *whenever* $m < n$

Proof. Since our renewals X_k, Y_k, Z_k are all non-negative *rv*'s, the point processes are always non-decreasing

$$S_m \leq S_{m+1} \qquad T_n \leq T_{n+1} \qquad m,n = 0,1,2,\dots$$

while

$$T_n = \frac{a\mu}{\lambda}S_n + \sum_{k=0}^{B(n)} Z_k \geq \frac{a\mu}{\lambda}S_n$$

Now, if $M(s) = m$ and $N(t) = n$, we must have both $S_m \leq s < S_{m+1}$ and $T_n \leq t < T_{n+1}$. Suppose now $0 \leq m < n$, namely $m+1 \leq n$ and $S_{m+1} \leq S_n$: then

$$\frac{a\mu}{\lambda}s < \frac{a\mu}{\lambda}S_{m+1} \leq \frac{a\mu}{\lambda}S_n \leq T_n \leq t$$

which apparently contradicts the hypothesized inequality. □

As a consequence when $a\mu s \geq \lambda t$ we can always restrict our calculations to the case $m \geq n \geq 0$. We can now finally state our complete results about the joint distributions $p_{m,n}(s,t)$.

Proposition 5.3. *Take for short*

$$y = \frac{\lambda t}{a} > 0 \qquad z = \frac{\lambda t - a\mu s}{a} < y$$

Then, when $a\mu s > \lambda t$, *namely* $z < 0$, *we have*

$$p_{m,n}(y,z) = \begin{cases} 0 & n > m \geq 0 \\ Q_{n,n}(y,z) & m = n \geq 0 \\ Q_{m,n}(y,z) - Q_{m,n+1}(y,z) & m > n \geq 0 \end{cases} \tag{5.3}$$

where we defined

$$Q_{m,n}(y,z) = a\int_0^y \mathrm{d}w\, h_n(aw)\sum_{k=n}^{m} \pi_{m-k}(w-z)\pi_k(y-w) \qquad m \geq n \geq 0 \tag{5.4}$$

When instead $a\mu s < \lambda t$, *and hence* $0 < z < y$, *we have*

$$p_{m,n}(y,z) = \begin{cases} A_{m,n}(y,z) - A_{m,n+1}(y,z) + B_{m,n}(y,z) - B_{m,n-1}(y,z) & n > m \geq 0 \\ A_{n,n}(y,z) - A_{n,n+1}(y,z) + B_{n,n}(y,z) + C_{n,n}(y,z) & m = n \geq 0 \\ A_{m,n}(y,z) - A_{m,n+1}(y,z) + C_{m,n}(y,z) - C_{m,n+1}(y,z) & m > n \geq 0 \end{cases} \tag{5.5}$$

where we defined

$$A_{m,n}(y,z) = a\int_0^z \mathrm{d}w\, h_n(aw)\pi_m(y-z) \qquad n,m \ge 0 \tag{5.6}$$

$$B_{m,n}(y,z) = a\int_0^z \mathrm{d}w\, h_{n+1}(aw)\sum_{k=0}^{n-m}\pi_k(z-w)\pi_m(y-z) \quad n \ge m \ge 0 \tag{5.7}$$

$$C_{m,n}(y,z) = a\int_z^y \mathrm{d}w\, h_n(aw)\sum_{k=n}^{m}\pi_{m-k}(w-z)\pi_k(y-w) \quad m \ge n \ge 1 \tag{5.8}$$

while for $m \ge n = 0$ we always have $C_{m,0}(y,z) = 0$. Moreover, both the results for $z < 0$ and for $0 < z < y$ connect with continuity in $z = 0$ in the sense that

$$p_{m,n}(y,0^-) = p_{m,n}(y,0^+) \qquad m,n \ge 0$$

Proof. Take first the case $a\mu s > \lambda t$, namely $z < 0$, and recall that for the integration variable in (5.2) it is $0 \le w \le y$. As a consequence, when the Proposition 5.1 in used in(5.2), we will always have

$$0 \le \tau = \frac{y-w}{\mu} \le \frac{y-z}{\mu} = s = \rho \tag{5.9}$$

On the other hand, since the conditions of the Proposition 5.2 are met, we can also restrict ourselves to evaluate $p_{m,n}(s,t)$ for $0 \le n \le m$. Then, by considering separately the cases $m = n \ge 0$ and $m > n \ge 0$, from (5.2) and from the Proposition 5.1 we first calculate $q_{m,n}, q_{m+1,n}, q_{m,n+1}$ and $q_{m+1,n+1}$, and finally (lengthy algebra is omitted [12]) from (5.1) we find (5.3).

When, on the other hand, $a\mu s < \lambda t$ (namely $y > z > 0$ and $0 < w < y$) and we use Proposition 5.1 in (5.2), instead of (5.9) we find

$$0 \le \tau = \frac{y-w}{\mu} \qquad 0 \le \rho = s = \frac{y-z}{\mu} \tag{5.10}$$

so that ρ and τ can now be in an order whatsoever. As a consequence, Proposition 5.2 does not hold, and we must consider all the possible orderings of m, n. Following then, the same line of reasoning as before, and always taking separately the different n, m orderings, a tedious calculation [12] gives first the q's from (5.2), and eventually the p's of our proposition from (5.1).

We finally show that the values of $p_{m,n}(y,z)$ separately listed in the Proposition 5.3 for $z < 0$ and $z > 0$ connect with continuity in $z = 0$, in the sense that for every $y > 0$

$$p_{m,n}(y,0^-) = p_{m,n}(y,0^+)$$

For $z < 0$ (namely $a\mu s > \lambda t$) the results are given in (5.3) and (5.4), so that for $z \uparrow 0^-$ and every $m, n \geq 0$, we simply have

$$p_{m,n}(y, 0^-) = 0 \qquad n > m \geq 0 \tag{5.11}$$

$$p_{n,n}(y, 0^-) = a\int_0^y \mathrm{d}w\, h_n(aw)\,\pi_0(w)\,\pi_n(y-w) \qquad n = m \geq 0 \tag{5.12}$$

$$\begin{aligned} p_{m,n}(y, 0^-) = a\int_0^y \mathrm{d}w\, \{&h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w) \qquad m > n \geq 0 \\ &+ [h_n(aw) - h_{n+1}(aw)] \textstyle\sum_{k=n}^m \pi_{m-k}(w)\pi_k(y-w)\} \end{aligned} \tag{5.13}$$

On the other hand, when $z > 0$ (namely $a\mu s < \lambda t$) we have (5.5), (5.6), (5.7) and (5.8), so that now z appears also as an integration limit, and some care should be exercised for $z \downarrow 0^+$. When indeed the integrand contains the distribution $\delta(x)$, as in fact happens in every first term of $h_n(x)$ which is $\beta_0(n)\delta(x) = a^n\delta(x)$ (see also Appendix A), we have for every regular function $\xi(x)$

$$\lim_{z\downarrow 0^+} \int_0^z \xi(x)\delta(x)\,\mathrm{d}x = \xi(0) \qquad\qquad \lim_{z\downarrow 0^+} \int_z^y \xi(x)\delta(x)\,\mathrm{d}x = 0.$$

As a consequence, we have

$$\lim_{z\downarrow 0^+} \int_0^z \mathrm{d}x\, \xi(x) h_n(x) = a^n \xi(0)$$

$$\lim_{z\downarrow 0^+} \int_z^y \mathrm{d}x\, \xi(x) h_n(x) = \int_0^y \mathrm{d}x\, \xi(x) \sum_{k=1}^n \beta_k(n) f_k(x).$$

With this provisions in mind, it is then only a question of sheer calculation [12] to show that the $p_{m,n}(y, z)$ for $z > 0$ as given in (5.5), (5.6), (5.7) and (5.8) correctly converge to the values (5.11), (5.12) and (5.13) for every possible ordering of n, m. For instance for $n > m \geq 0$, with $x = aw$ and recalling also that $\pi_k(0) = \delta_{k,0}$ (so that $\pi_{n-m}(0) = 0$ because $n > m$), in the limit $z \downarrow 0^+$, we immediately have

$$\begin{aligned} &p_{m,n}(y, 0^+) \\ &= \pi_m(y)\int_0^{0^+} \mathrm{d}x \left[a^n - a^{n+1} + a^n\pi_{n-m}(0) - (a^n - a^{n+1})\sum_{k=0}^{n-m}\pi_k(0)\right]\delta(x) \\ &= \pi_m(y)\left[a^n - a^{n+1} - (a^n - a^{n+1})\sum_{k=0}^{n-m}\delta_{k,0}\right] = 0 \end{aligned}$$

and so on for the other two cases. □

Proposition 5.4. *The terms Q, A, B and C in the Proposition 5.3 can be expressed in terms of finite combinations of elementary functions: when $z < 0$ (namely $a\mu s > \lambda t$), we have for $m \geq n \geq 0$*

$$Q_{m,n}(y,z) = \sum_{k=n}^{m}\sum_{j=k}^{m}\frac{(-1)^{j-k}}{a^j}\binom{j}{k}\sum_{\ell=0}^{n}\beta_\ell(n)$$
$$\pi_{m-j}(y-z)\pi_{j+\ell}(ay)\,\Phi(j+1;j+\ell+1;ay) \qquad (5.14)$$

Here and in the following $\Phi(\alpha;\beta;x)$ *are confluent hypergeometric functions. When instead* $z > 0$ *(namely* $a\mu s < \lambda t$*), we have for every* $n, m \geq 0$

$$A_{m,n}(y,z) = \pi_m(y-z)\sum_{k=0}^{n}\beta_k(n)\left[1 + \pi_k(az) - \sum_{j=0}^{k}\pi_j(az)\right] \qquad (5.15)$$

while for $n \geq m \geq 0$ *it is*

$$B_{m,n}(y,z) = \pi_m(y-z)\sum_{k=0}^{n-m}\pi_k\,(z)\sum_{\ell=0}^{n+1}\beta_\ell(n+1) \qquad (5.16)$$
$$\times\frac{(az)^\ell k!}{(k+\ell)!}\,\Phi\left(\ell, k+\ell+1, (1-a)z\right)$$

and for $m \geq n \geq 1$ *(for* $m \geq n = 0$ *we always have* $C_{m,0}(y,z) = 0$*), it is*

$$C_{m,n}(y,z) = \frac{e^{-(1-a)(y-z)}}{a^m}\sum_{\ell=1}^{n}\beta_\ell(n)\sum_{k=n}^{m}\sum_{j=0}^{\ell-1}(-1)^{\ell-1-j}\binom{k+\ell-j-1}{k}$$
$$\pi_j(ay)\pi_{m+\ell-j}(a(y-z))\Phi(k+\ell-j, m+\ell-j+1, a(y-z)) \qquad (5.17)$$

Finally, since the parameters α, β *of the* $\Phi(\alpha,\beta,x)$ *involved in the previous equations are integer numbers with* $0 \leq \alpha < \beta$*, our confluent hypergeometric functions are just finite combinations of powers and exponentials according to the following formulas:*

$$\Phi(0\,,\beta,x) = 1 \qquad \beta > \alpha = 0$$
$$\Phi(\alpha,\beta,x) = e^x\sum_{\gamma=1}^{\alpha}(-1)^{\alpha-\gamma}\binom{\beta-\gamma-1}{\beta-\alpha-1}\frac{\pi_{\gamma-1}(x)}{\pi_{\beta-1}(x)}\,\Pi_{\alpha-\gamma+1}(x) \qquad \beta > \alpha \geq 1$$

Proof. The detailed proof unfolds along a sequence of integrations based on known results and is here omitted for the sake of brevity [12]. □

We end this section with a short list of a few explicit examples of joint probabilities holding in the region $a\mu s \geq \lambda t$:

$$
\begin{aligned}
p_{0,0}(s,t) &= e^{-\mu s} \\
p_{1,0}(s,t) &= \frac{e^{-\mu s}}{a}\left[(1-a)(1-e^{-\lambda t}) + a\mu s - \lambda t\right] \\
p_{1,1}(s,t) &= \frac{e^{-\mu s}}{a}\left[\lambda t - (1-a)(1-e^{-\lambda t})\right] \\
p_{2,0}(s,t) &= \frac{e^{-\mu s}}{2a^2}\left[2(1-a)(1+a\mu s)(1-e^{-\lambda t}) + (a\mu s - \lambda t)^2 - 2(1-a)\lambda t\right] \\
p_{2,1}(s,t) &= \frac{e^{-\mu s}}{a^2}\left[(1-a)(a-4-(1-a)\lambda t - a\mu s)(1-e^{-\lambda t})\right. \\
&\quad \left. + \lambda t(a^2 - 5a + 4 + a\mu s - \lambda t)\right] \\
p_{2,2}(s,t) &= \frac{e^{-\mu s}}{2a^2}\left[2(1-a)(3-a+(1-a)\lambda t)(1-e^{-\lambda t})\right. \\
&\quad \left. + \lambda t(\lambda t - 2(1-a)(3-a))\right]
\end{aligned}
$$

6. Cross-Correlations and Relative Timing

In this section, we will briefly discuss the main correlation properties of the two processes. We first of all look at the point processes and we remark that, by recasting the self-decomposability equation (4.1) in the form

$$X_k = \frac{a\mu}{\lambda} W_k + B_k(1) Z_k \qquad W_k = \frac{\lambda}{\mu} Y_k \sim \mathfrak{E}_1(\mu)$$

the point processes appear as

$$T_n = \sum_{k=0}^{n} X_k \qquad S_n = \sum_{k=0}^{n} W_k$$

where now $X_k \sim \mathfrak{E}_1(\lambda)$ and $W_k \sim \mathfrak{E}_1(\mu)$ play at once the role of the correlated renewals. It is interesting to point out then that, at variance with other models [7], we are no longer tied to take truly *coincident* shocks: we will show indeed that with non-zero probabilities the values of the paired and correlated renewals X_k, W_k (waiting times) can be in an order whatsoever, and they would almost never coincide. As a consequence, the propagation of the shocks from a process to the other will quite plausibly happen with delays whose random sizes (and directions) could also be modeled by suitably choosing our parameters a, λ and μ. And moreover the random times T_n and S_n will be correlated by the summing up of the renewals, but will never fall at the same instant. This *relative timing* apparently allows for an enhanced flexibility of the model in the practical applications, because we no longer have to rely on common shocks, but rather on correlated and randomly delayed ones.

More precisely, we can now single out two possible regimes for our processes: ${}^{a\mu}/_{\lambda} \leq 1$ and ${}^{a\mu}/_{\lambda} > 1$. It is then easy to see that for every $k = 1, 2, \ldots$

$$X_k = \frac{a\mu}{\lambda} W_k + B_k(1) Z_k \geq \frac{a\mu}{\lambda} W_k > W_k \qquad \frac{a\mu}{\lambda} > 1$$

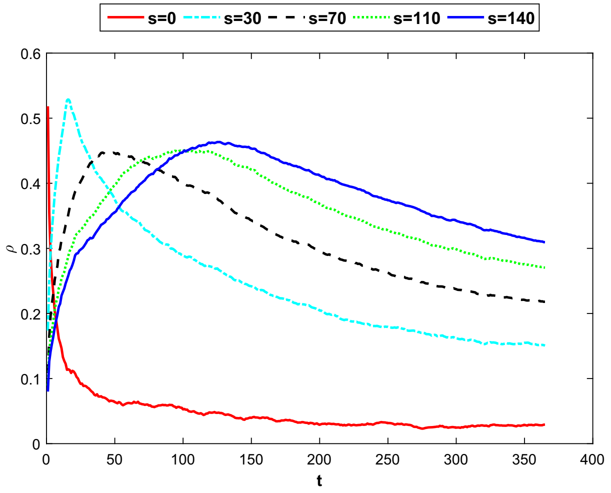

Figure 6. Cross-correlation $\rho(s,t)$ of the two Poisson processes $M(s)$ and $N(t)$ estimated by Monte Carlo simulations. Here, $a = 0.5$, and $\lambda = \mu = 20$, namely ${}^{a\mu}/_{\lambda} = 0.5 < 1$

and, hence, we first of all have

$$\boldsymbol{P}\{X_k > W_k\} = 1 \qquad \frac{a\mu}{\lambda} > 1$$

On the other hand for ${}^{a\mu}/_{\lambda} \leq 1$, the probability $\boldsymbol{P}\{X_k > W_k\}$ can still be explicitly calculated by taking into account the laws specified in the Sect. 4, and in this case it is possible to show that

$$\boldsymbol{P}\{X_k > W_k\} = \frac{(1-a)\mu}{\lambda + (1-a)\mu} \qquad \frac{a\mu}{\lambda} \leq 1$$

a value ranging from 0 to 1 according to the different possible choices of the parameters a, λ and μ.

As for the relative timings T_n, S_m of the shocks along the point processes themselves, an explicit calculation of $\boldsymbol{P}\{T_n \leq S_m\}$ is certainly possible, but its results would turn out to be rather cumbersome because it would involve two or three convolutions of (positive and negative) Erlang laws with different parameters. We will then confine ourselves here to produce just the cross-correlations between T_n, S_m: since it is easy to check that

$$\boldsymbol{cov}[X_k, W_\ell] = \frac{a}{\lambda\mu}\,\delta_{k\ell}$$

it is also apparent that

$$\boldsymbol{cov}[T_n, S_m] = \sum_{k=1}^{n}\sum_{\ell=1}^{m}\boldsymbol{cov}[X_k, W_\ell] = \frac{a}{\lambda\mu}\sum_{k=1}^{n}\sum_{\ell=1}^{m}\delta_{k\ell} = \frac{a}{\lambda\mu}m \wedge n$$

and, hence, the cross-correlation coefficient of T_n, S_m will simply be

$$r_{nm} = a\,\frac{n \wedge m}{\sqrt{nm}} = \begin{cases} a\sqrt{{}^{n}/_{m}} & \text{for } \ n \leq m \\ a\sqrt{{}^{m}/_{n}} & \text{for } \ n \geq m \end{cases}$$

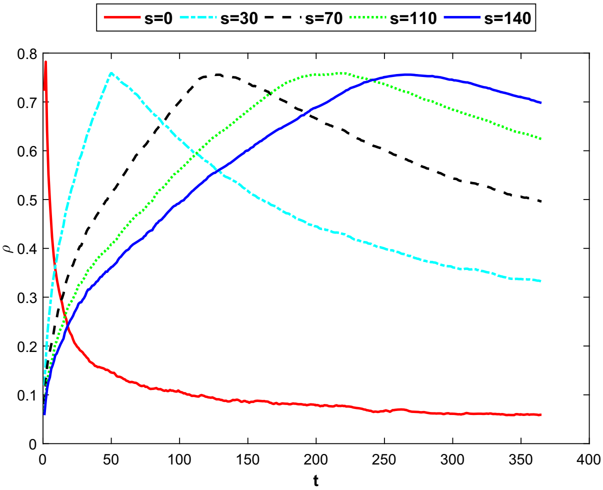

Figure 7. Cross-correlation $\rho(s,t)$ of the two Poisson processes $M(s)$ and $N(t)$ estimated by Monte Carlo simulations. Here, $a = 0.8$, $\lambda = 20$ and $\mu = 40$, namely ${}^{a\mu}/_{\lambda} = 1.6 > 1$

Finally, even closed formulas for the cross-correlation coefficient $\rho(s,t)$ between the Poisson processes $M(s)$ and $N(t)$ are still derivable on the ground of our previous results about the joint distributions, but it would be too long to thoroughly elaborate them here. As an alternative we have chosen to show the plots of their estimates based on a sample of 10^5 Monte Carlo simulations of their trajectories as shown in the Figs. 6 and 7. There the behavior is displayed of $\rho(s,t)$ as a function of t for different, fixed values of s. More precisely, in Fig. 6 we have taken $a = 0.5$ and $\lambda = \mu = 20$ as the values for the relevant parameters of our coupled processes (then we have ${}^{a\mu}/_{\lambda} < 1$), while in the Fig. 7 the same parameters are $a = 0.8, \lambda = 20$ and $\mu = 40$ (and then ${}^{a\mu}/_{\lambda} > 1$). It is apparent from these pictures that the behavior of $\rho(s,t)$ is comparable to that of the self-correlation of a simple Poisson process, but for the fact that the cumulative effect of the correlate renewals produces a smoothing of the shape around the maximum values near $t = s$. At first sight, this could look as a little difference, but in the domain, for instance, of the financial applications even small deviations among the models could produce huge differences in gains and losses.

7. Conclusions and Further Inquiries

It is apparent that, within the model discussed in the Sect. 2, from the self-decomposability alone we can only get pairs of *rv*'s X, Y with *positive correlations* $0 < r_{XY} < 1$ steered by the value of a parameter a. It would be interesting, however, to widen the scope of our models to achieve also Poisson processes whose correlation can span over all its possible values (both positive and negative) by changing the value of some numerical parameter.

In this respect, it is important to remark—as pointed out in the Sect. 3.3—that while two *rv*'s X and Y which are, for instance, marginally exponentials can also be *totally correlated* ($r_{XY} = 1$), they cannot instead be *totally anti-correlated* ($r_{XY} = -1$) because this would imply some *linear* dependence with a *negative* proportionality coefficient, and that would be at odds with the fact that both our *rv*'s take arbitrary large, but only positive values. Hence, two exponential *rv*'s X and Y can always have a negative correlation, but only up to a minimal value which in any case must be larger than -1. We also showed in Sect. 3.3 that this minimum is reached; when between X and Y there is a peculiar kind of mutual functional, *decreasing* dependence, albeit clearly *not a linear* one. A model to produce pairs X, Y of *rv*'s which are marginally exponentials, and which—following the value of a numerical parameter a—show all the possible correlation values will be discussed in a subsequent paper.

Our results in any case show that the self-decomposability, joined with the subordination techniques, can be a promising tool to study dependency beyond the Gaussian-Itô world. We have shown indeed how to obtain dependent exponential (gamma) *rv*'s that can be used to create and simulate dependent Poisson processes without resorting to definitely coincident jumps, but the path is now open to produce more general dependent gamma (Erlang at first) *rv*'s to simulate dependent variance gamma processes. A further extension could then be to study the self-decomposability of density functions that have a strictly proper rational characteristic function (Exponential Polynomial Trigonometric, *EPT* laws) in order to construct 2-dimensional correlated *EPT rv*'s (see for instance [20,21]). Finally, it would be expedient to explore the Markov properties of the 2-component Poisson processes $(M,(t), N(t))$ with dependent marginals that we have introduced in this paper and the Master equations ruling them: this too will be the subject of future inquiries.

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Appendix A: Notations

All along this paper we will adopt the following notations: for a Poisson law $\mathfrak{P}(\alpha)$, we will introduce the symbols

$$\pi_n(\alpha) = \frac{\alpha^n}{n!}\, e^{-\alpha} \qquad \Pi_n(\alpha) = \sum_{k=n}^{\infty} \pi_k(\alpha) \qquad \alpha > 0 \qquad n = 0, 1, 2 \ldots$$

and for a binomial law $\mathfrak{B}(n, 1-a)$ the notation

$$\beta_k(n) = \begin{cases} 1 & n = 0, \qquad k = 0 \\ \binom{n}{k} a^{n-k}(1-a)^k & n = 1, 2, \ldots, k = 0, \ldots, n \end{cases} \qquad 0 \le a \le 1$$

It will be understood moreover that

$$\pi_n(0^+) = \delta_{n,0}$$

We will also use but the Heaviside function ϑ, and the Heaviside symbol Θ

$$\vartheta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \Theta_j = \begin{cases} 1 & j \geq 1 \\ 0 & j \leq 0 \end{cases} \qquad j = 0, \pm 1, \pm 2, \dots$$

The *pdf* and *chf* of a standard Erlang law $\mathfrak{E}_n(1)$ moreover will be denoted as:

$$f_n(x) = \begin{cases} \delta(x) \\ \frac{x^{n-1}}{(n-1)!} e^{-x} \vartheta(x) \end{cases} \qquad \varphi_n(u) = \begin{cases} 1 \\ \left(\frac{1}{1-iu}\right)^n \end{cases} \qquad \begin{matrix} n = 0 \\ n = 1, 2, \dots \end{matrix}$$

where it is understood for the Dirac delta $\delta(x)$ that for every $b > 0$

$$\int_0^b \delta(x)\,\mathrm{d}x = 1 \qquad \lim_{z \downarrow 0^+} \int_z^b \delta(x)\,\mathrm{d}x = 0$$

Remark also that apparently

$$f_n(x) = \pi_{n-1}(x)\vartheta(x) \qquad n = 1, 2, \dots$$

We will finally define for later convenience the functions

$$h_n(x) = \sum_{k=0}^{n} \beta_k(n) f_k(x) \qquad n = 0, 1, 2, \dots$$

which are the *pdf*'s of the mixtures of Erlang laws $\mathfrak{E}_k(1)$ with binomial $\mathfrak{B}(n, 1-a)$ weights for their indices k.

Appendix B: A Proof of Proposition 5.1

To evaluate $\boldsymbol{P}\{S_m \leq \rho,\, S_n \leq \tau\}$, we first remark that

$$\boldsymbol{P}\{S_m \leq \rho,\, S_n \leq \tau\} = \boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\}$$

and then that, being a Poisson process, $M(t)$ is non-decreasing: as a consequence

$$\begin{aligned} m \leq n \text{ and } \tau \leq \rho &\Longrightarrow M(\tau) \leq M(\rho) \quad \text{hence } \{M(\tau) \geq n\} \subseteq \{M(\rho) \geq m\} \\ n \leq m \text{ and } \rho \leq \tau &\Longrightarrow M(\rho) \leq M(\tau) \quad \text{hence } \{M(\rho) \geq m\} \subseteq \{M(\tau) \geq n\} \end{aligned}$$

In the case $m \leq n$ we then have for $\tau \leq \rho$

$$\boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\} = \boldsymbol{P}\{M(\tau) \geq n\}$$

while for $\rho \leq \tau$ from the general properties of a Poisson process, we get

$$\begin{aligned}
&\boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\} \\
&\quad = \sum_{k=m}^{\infty} \boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n \,|\, M(\rho) = k\}\, \boldsymbol{P}\{M(\rho) = k\} \\
&\quad = \sum_{k=m}^{n} \boldsymbol{P}\{M(\tau) \geq n \,|\, M(\rho) = k\}\, \boldsymbol{P}\{M(\rho) = k\} + \boldsymbol{P}\{M(\rho) > n\} \\
&\quad = \sum_{k=m}^{n} \boldsymbol{P}\{M(\tau - \rho) \geq n - k\}\, \boldsymbol{P}\{M(\rho) = k\} + \boldsymbol{P}\{M(\rho) > n\}
\end{aligned}$$

In the same vein, when $n \leq m$ we have for $\rho \leq \tau$

$$\boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\} = \boldsymbol{P}\{M(\rho) \geq n\}$$

while for $\tau \leq \rho$ we get

$$\begin{aligned}
&\boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\} \\
&\quad = \sum_{k=m}^{\infty} \boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n \,|\, M(\tau) = k\}\, \boldsymbol{P}\{M(\tau) = k\} \\
&\quad = \sum_{k=n}^{m} \boldsymbol{P}\{M(\rho) \geq m \,|\, M(\tau) = k\}\, \boldsymbol{P}\{M(\tau) = k\} + \boldsymbol{P}\{M(\tau) > m\} \\
&\quad = \sum_{k=n}^{m} \boldsymbol{P}\{M(\rho - \tau) \geq m - k\}\, \boldsymbol{P}\{M(\tau) = k\} + \boldsymbol{P}\{M(\tau) > m\}
\end{aligned}$$

Remark that for $m = n$ both the cases lead to the same result, namely

$$\boldsymbol{P}\{M(\rho) \geq n,\, M(\tau) \geq n\} = \begin{cases} \boldsymbol{P}\{M(\tau) \geq n\} & \text{when } \tau \leq \rho \\ \boldsymbol{P}\{M(\rho) \geq n\} & \text{when } \rho \leq \tau \end{cases}$$

that can also be conveniently summarized as:

$$\boldsymbol{P}\{M(\rho) \geq n,\, M(\tau) \geq n\} = \boldsymbol{P}\{M(\rho \wedge \tau) \geq n\}$$

On the other hand for $m < n$, we have

$$\begin{aligned}
&\text{for } \tau \leq \rho \quad \boldsymbol{P}\{M(\tau) \geq n\} \\
&\text{for } \rho \leq \tau \quad \boldsymbol{P}\{M(\rho) \geq n\} + \sum_{k=m}^{n-1} \boldsymbol{P}\{M(\tau - \rho) \geq n - k\}\, \boldsymbol{P}\{M(\rho) = k\}
\end{aligned}$$

that can also be put in the form

$$\boldsymbol{P}\{M(\rho \wedge \tau) \geq n\} + \vartheta(\tau - \rho) \sum_{k=m}^{n-1} \boldsymbol{P}\{M(\tau - \rho) \geq n - k\}\, \boldsymbol{P}\{M(\rho) = k\}$$

while for $m > n$ it is

$$\begin{aligned}
&\text{for } \tau \leq \rho \quad \boldsymbol{P}\{M(\tau) \geq m\} + \sum_{k=n}^{m-1} \boldsymbol{P}\{M(\rho - \tau) \geq m - k\}\, \boldsymbol{P}\{M(\tau) = k\} \\
&\text{for } \rho \leq \tau \quad \boldsymbol{P}\{M(\rho) \geq m\}
\end{aligned}$$

namely

$$\boldsymbol{P}\left\{M(\rho\wedge\tau)\geq m\right\}+\vartheta(\rho-\tau)\sum_{k=n}^{m-1}\boldsymbol{P}\left\{M(\rho-\tau)\geq m-k\right\}\boldsymbol{P}\left\{M(\tau)=k\right\}$$

In both cases, the first terms can expressed as $\boldsymbol{P}\left\{M(\rho\wedge\tau)\geq m\vee n\right\}$, and in this form they also coincide with the previous result for $m=n$. On the other hand, the extra term with the sum (which is absent for $m=n$) must be taken in consideration either when we have both $m<n$ and $\rho\leq\tau$, or when it is $m>n$ and $\tau\leq\rho$. All these provisions can then be comprehensively taken into account in the formula

$$\begin{aligned}&\boldsymbol{P}\left\{S_m\leq\rho,\,S_n\leq\tau\right\}\\&\quad=\boldsymbol{P}\left\{M(\rho\wedge\tau)\geq m\vee n\right\}+\left[\Theta_{n-m}\vartheta(\tau-\rho)+\Theta_{m-n}\vartheta(\rho-\tau)\right]\\&\qquad\cdot\sum_{k=m\wedge n}^{(m\vee n)-1}\boldsymbol{P}\left\{M(|\rho-\tau|)\geq(m\vee n)-k\right\}\boldsymbol{P}\left\{M(\rho\wedge\tau)=k\right\}\end{aligned}$$

which finally takes the form of Proposition 5.1 using the notations adopted in the Appendix A for the Poisson distributions and the Heaviside symbols.

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