Phenomenology from relativistic Lévy–Schrödinger equations: Application to neutrinos

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A close connection between Feynman propagators and a particular Lévy stochastic process is established. The approach can be easily applied to the Standard Model $SU_C(3) \times SU_L(2) \times U(1)$ providing interesting, qualitative results. Quantitative results, compatible with experimental data, are obtained in the case of neutrinos.

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I. INTRODUCTION

In a previous note [1] we showed that it is possible to derive and modify the relativistic Feynman propagator of a free (forceless) particle (fermion spin $\frac{1}{2}$, boson spin 0 and 1) on the basis of Lévy stochastic processes [2]. We adopt here the space-time relativistic approach of Feynman's propagators (for bosons and fermions) instead of the canonical Lagrangian-Hamiltonian quantized field theory. The rationale for this choice is that for the development of our basic ideas the former alternative is better suited to exhibit the connection between the propagator of quantum mechanics and the underlying Lévy processes. More precisely, the relativistic Feynman propagators are here linked to a dynamical theory based on a particular Lévy process: a point, already discussed in a previous paper [3], which is here analyzed thoroughly with the purpose of deducing its consequences for the basic interactions among the fundamental constituents, namely quarks, leptons, gluons, photons and so on.

A stochastic process $X(t)$, $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lévy process if $X(0) = 0$, $\mathbb{P}$-qo, if it has independent and stationary increments, and if it is stochastically continuous. To simplify the notation, in this introduction we will restrict ourselves only to one-dimensional processes, but the three-dimensional extension is straightforward and will be adopted in the subsequent sections. It is well known [2, 4, 5] that all its laws are infinitely divisible, but we will be mainly interested in the non stable (and in particular non Gaussian) case[1]. In this case the characteristic functions of the process increments are $[\varphi(u)]^{\Delta t/\tau}$ where φ is infinitely divisible, but not stable, and τ is a time scale. The transition probability density of a particle moving from the space-time point 1 to 2 then is

$$p(2|1) = p(x_2, t_2|x_1, t_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du\, [\varphi(u)]^{(t_2-t_1)/\tau} e^{-iu(x_2-x_1)} \tag{1}$$

and, in analogy with the non relativistic Wiener case, the Feynman propagator of a free particle is

$$\mathcal{K}(2|1) = \mathcal{K}(x_2, t_2|x_1, t_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du\, [\varphi(u)]^{i(t_2-t_1)/\tau} e^{-iu(x_2-x_1)} \tag{2}$$

so that the wave function evolution is

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx'\, \mathcal{K}(x,t|x',t')\psi(x',t'). \tag{3}$$

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[1] A law φ is said to be *infinitely divisible* if for every n it exists a characteristic function φ_n such that $\varphi = \varphi_n^n$; on the other hand it is said to be *stable* when for every $c > 0$ it is always possible to find $a > 0$ and $b \in \mathbf{R}$ such that $e^{ibu}\varphi(au) = [\varphi(u)]^c$. Every stable law is also infinitely divisible.

From (2) and (3) we easily obtain [3]

$$i\partial_t\psi = -\frac{1}{\tau}\eta(\partial_x)\psi$$

where $\eta = \log\varphi$ and $\eta(\partial_x)$ is a pseudodifferential operator with symbol $\eta(u)$ which plays the role of the infinitesimal generator of the process semigroup $T_t = e^{t\eta(\partial_x)/\tau}$ [2, 6–8].

It is well known [2, 4] that φ is infinitely divisible if and only if $\eta(u) = \log\varphi(u)$ satisfies the Lévy–Khintchin formula

$$\eta(u) = i\gamma u - \frac{\beta^2u^2}{2} + \int_{\mathbb{R}}\left[e^{iux} - 1 - iux\, I_{[-1,1]}(x)\right]\,\nu(dx) \tag{4}$$

where $\gamma, \beta \in \mathbb{R}$, $I_{[-1,1]}(x)$ is the indicator of $[-1, 1]$, and $\nu(dx)$ is a Lévy measure. Then, in the most common cases of centered and symmetric laws the equation (4) simplifies in

$$\eta(u) = -\frac{\beta^2u^2}{2} + \int_{\mathbb{R}}(\cos ux - 1)\,\nu(dx) \tag{5}$$

and $\eta(u)$ becomes even and real. As a consequence the corresponding operator $\eta(\partial_x)$ is self–adjoint and acts on propagators and wave functions according to the Lévy–Schrödinger integro–differential equation

$$i\partial_t\psi(x,t) = -\frac{1}{\tau}\,\eta(\partial_x)\psi(x,t) = -\frac{\beta^2}{2\tau}\,\partial_x^2\psi(x,t) - \frac{1}{\tau}\int_{\mathbb{R}}\left[\psi(x+y,t) - \psi(x,t)\right]\,\nu(dy) \tag{6}$$

The free equation (6) admits simple stationary solutions: if we take $\psi(x,t) = e^{-iE_0t/\alpha}\phi(x)$ and $\alpha = m\beta^2/\tau$, we get

$$E_0\phi(x) = -\frac{\alpha^2}{2m}\,\phi''(x) - \frac{\alpha}{\tau}\int_{\mathbb{R}}[\phi(x+y) - \phi(x)]\,\nu(dy), \tag{7}$$

so that for a plane wave $\phi(x) = e^{\pm iux}$, and for a symmetric Lévy noise, from (5) we have

$$E_0\phi(x) = -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}}\left(e^{\pm iuy} - 1\right)\,\nu(dy)\right]e^{\pm iux} = -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}}(\cos uy - 1)\,\nu(dy)\right]\phi(x) = -\frac{\alpha}{\tau}\eta(u)\phi(x)$$

which entails $E_0 = -\alpha\eta(u)/\tau$. Then, switching back to three dimensions, and introducing the momentum $\boldsymbol{p} = \alpha\boldsymbol{u}$, we obtain [3] the relevant equation

$$E_0 = -\frac{\alpha}{\tau}\,\eta(\boldsymbol{u}) = -\frac{\alpha}{\tau}\,\eta\left(\frac{\boldsymbol{p}}{\alpha}\right) \tag{8}$$

which gives the model for the energy-momentum relations that we will use in the following.

II. EXTENDED RELATIVISTIC QUANTUM EQUATIONS

Starting from (8) the particular non stable law (for details see for example [2])

$$\eta(\boldsymbol{u}) = 1 - \sqrt{1 + a^2\boldsymbol{u}^2} \tag{9}$$

(from now on we will write $\boldsymbol{u}^2$ instead of $|\boldsymbol{u}^2|$) with the following identification of the parameters

$$a = \frac{\hbar}{mc} \qquad \alpha = \hbar \qquad \tau = \frac{\hbar}{mc^2}$$

will lead to the formula

$$E_0 = -mc^2\eta\left(\frac{\boldsymbol{p}}{\hbar}\right) = E - mc^2 = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2} - mc^2 \tag{10}$$

which is the well–known relativistic kinetic energy for a particle of mass m. The Schrödinger equation of a relativistic free-particle is then easily obtained from (10) by reinterpreting as usual E and $\boldsymbol{p}$ respectively as the operators $i\hbar\partial_t$ and $-i\hbar\boldsymbol{\nabla}$

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - \hbar^2c^2\boldsymbol{\nabla}^2}\,\psi(x,t) \tag{11}$$

It has been shown [2, 9, 10] that the Lévy process with logarithmic characteristic (9) which is behind the equation (11) is a pure jump process with an absolutely continuous Lévy measure $\nu(d^3\boldsymbol{x}) = W(\boldsymbol{x})\,d^3\boldsymbol{x}$, where [2, 3]

$$W(\boldsymbol{x}) = \frac{1}{2a\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{|\boldsymbol{x}|}{a}\right) = \frac{mc}{2\hbar\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{x}|\right) \tag{12}$$

(K_ν are the modified Bessel functions [11]), so that from (**??**) the equation (11) takes the integro-differential form

$$i\hbar\partial_t\psi(\boldsymbol{x},t) = -mc^2\int_{\mathbb{R}^3}\frac{\psi(\boldsymbol{x}+\boldsymbol{y},t)-\psi(\boldsymbol{x},t)}{2\pi^2\boldsymbol{y}^2}\,\frac{mc}{\hbar}K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{y}|\right)d^3\boldsymbol{y} \tag{13}$$

Of course from (11) one also derives the free Klein–Gordon and Dirac equations both for scalar, and for spinor wave functions [12]

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi \;=\; 0, \tag{14}$$

$$\left(i\gamma_\mu\partial^\mu - \frac{mc}{\hbar}\right)\psi \;=\; 0. \tag{15}$$

while the respective propagators satisfy the inhomogeneous equations

$$\left(\Box_2 - m^2\right)\mathcal{K}_{KG}(2|1) \;=\; \delta^4(2|1) \tag{16}$$

$$(i\gamma_\mu\partial_2^\mu - m)\,\mathcal{K}_D(2|1) \;=\; i\,\delta^4(2|1) \tag{17}$$

with $\hbar = c = 1$, and $\delta^4(2|1) = \delta(t_2-t_1)\delta^3(\boldsymbol{x}_2-\boldsymbol{x}_1)$. Let us finally remark that these relativistic quantum wave equations have been even recently of particular interest [13, 14] also in the field of quantum optical phenomena and of quantum information.

We consider now a class of transformations both of $\eta(\boldsymbol{u})$ in (9), and of the corresponding relativistic total energy

$$E(\boldsymbol{p}) = mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}} \tag{18}$$

by requiring on the one hand that they preserve the infinite divisibility (so that our model will always be based on a suitable Lévy process), and on the other that they modify the free equations of motion (14) and (15) in such a way that in the subsequent sections we will be able to eliminate the usual field theoretical divergencies and obtain a mass spectrum. To this end we propose to extend the energy-momentum formula (18) in the following way

$$E(\boldsymbol{p}) = mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}+f\left(\frac{p^2}{m^2c^2}\right)} \tag{19}$$

where f is a dimensionless, smooth function of the relativistic scalar p^2/m^2c^2 (here $p^2 = E^2/c^2 - \boldsymbol{p}^2$). Of course this modification entails that p^2 no longer coincides with m^2c^2 since the standard energy-momentum relation is now changed into

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2 + m^2c^2 f\left(\frac{p^2}{m^2c^2}\right). \tag{20}$$

As we will see in the following, this also implies that the mass no longer is m: it will take instead one or more values depending on the choice of f. As a matter of fact, it could appear preposterous to introduce a function f of an argument which after all is a constant (albeit different from 1), but this artifice – while keeping a viable connection to a suitable underlying Lévy process – will lend us the possibility of having both a mass spectrum, and a new wave equation when we quantize our classical relations.

To see that, we first remark that (20) defines the total particle energy E only in an implicit form. To find it explicitly we just rewrite (20) as

$$g\left(\frac{p^2}{m^2c^2}\right) = \frac{p^2}{m^2c^2} - f\left(\frac{p^2}{m^2c^2}\right) = 1. \tag{21}$$

with $g(x) = x - f(x)$, so that $x = p^2/m^2c^2$ must be one of the (possibly many) solutions of $g(x) = x - f(x) = 1$. Remark that if we require that $x = 1$ (namely $p^2 = m^2c^2$) is a solution, then we must have $f(1) = 0$ and $g(1) = 1$. If now $g^{-1}(1)$ represents one of the said solutions, we will have $p^2/m^2c^2 = g^{-1}(1)$ so that

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2g^{-1}(1)$$

which can first of all be interpreted as a simple mass re-scaling, from m to one of the (possibly many) values $M = m\sqrt{g^{-1}(1)}$. The new hamiltonian then is

$$E(\boldsymbol{p}) = \sqrt{m^2c^4g^{-1}(1) + \boldsymbol{p}^2c^2} = Mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{M^2c^2}} \tag{22}$$

and this mass re-scaling $m \to M$ has as its first straightforward consequence that the associated logarithmic characteristic η underwent little changes, so that it is again trivially infinitely divisible (albeit with different numerical parameters) and hence still produces acceptable Lévy processes. But there is more: since $g^{-1}(1)$ can take several different real and positive values, by means of our extension (19) we can introduce a mass spectrum as we will see in the following.

From the modified energy formula (19) – by using the prescriptions $E \to i\hbar\partial_t$ and $\boldsymbol{p} \to -i\hbar\boldsymbol{\nabla}$ – one now derives the extended relativistic Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = mc^2\sqrt{1 - \frac{\hbar^2\boldsymbol{\nabla}^2}{m^2c^2} + f\left(\frac{\hbar^2\Box}{m^2c^2}\right)}\,\psi(x,t) \tag{23}$$

and then in the usual way one can achieve the modified Klein-Gordon and Dirac equations ($\hbar = c = 1$)

$$\left[\Box - m^2 f\left(\frac{\Box}{m^2}\right) - m^2\right]\psi \;=\; 0 \tag{24}$$

$$\left[i\gamma_\mu\partial^\mu - m\sqrt{1 + f\left(\frac{\Box}{m^2}\right)}\right]\psi \;=\; 0 \tag{25}$$

whereas the corresponding Feynman propagators verify the inhomogeneous equations

$$\left[\Box_2 - m^2 f\left(\frac{\Box_2}{m^2}\right) - m^2\right]\mathcal{K}_{KG}(2|1) \;=\; \delta^4(2|1) \tag{26}$$

$$\left[i\gamma_\mu\partial_2^\mu - m\sqrt{1 + f\left(\frac{\Box_2}{m^2}\right)}\right]\mathcal{K}_D(2|1) \;=\; i\delta^4(2|1) \tag{27}$$

In momentum space[2] they have the following representation

$$\mathcal{K}_{KG}(p^2) \;=\; \frac{1}{p^2 - m^2\left[1 + f(p^2/m^2)\right] + i\epsilon} \tag{28}$$

$$\mathcal{K}_D(p^2) \;=\; \frac{1}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} \tag{29}$$

We finally remark that, in absence of interaction, the equations (26), (27), (28) and (29) will go back to the well known, usual formulae when $f(x) \to 0$.

[2] The connections between the wave functions and their propagators of the equations (26) and (27) are then

$$\psi(2) \;=\; \int\left[\psi(1)\partial_{1\mu}\mathcal{K}_{KG}(2|1) - \partial_{1\mu}\psi(1)\mathcal{K}_{KG}(2|1)\right]N^\mu(1)\,d^3V_1$$

$$\psi(2) \;=\; \int\mathcal{K}_D(2|1)\gamma^\mu N_\mu(1)\psi(1)\,d^3V_1$$

$N_\mu(1)$ being the inward normal to the surface at the point 1.

III. PHENOMENOLOGY

Let us now consider the Feynman perturbative contributions in renormalized field theories and focus our attention on the self-energy terms in QED, QCD and in SM $SU_C(3) \times SU_L(2) \times U(1)$. The amplitude for a fermion that propagates from the vertex X to Y, if expanded, looks as follows (the numerator is not essential for our purposes)

$$A = A^{(0)} + A^{(1)} + A^{(2)} + \dots \tag{30}$$

with zero order

$$A^{(0)} = Y \frac{1}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} X \tag{31}$$

If the fermion emits and reabsorbs a virtual boson with mass M ($M = M_Z, M_{W^\pm}$ for weak interactions, $M = 0$ for photons or gluons) we have

$$A^{(1)} = Y\gamma^\rho \frac{1}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} C \frac{1}{\gamma^\nu p_\nu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} \gamma_\rho X \tag{32}$$

where

$$C = \int \frac{4\pi g_s^2\, d^4k}{[(p-k)^2 - M^2][\gamma^\mu k_\mu - m\sqrt{1 + f\,(k^2/m^2)}] + i\epsilon} = \tilde{A}(p^2)\,\gamma\cdot p + \tilde{B}(p^2) \tag{33}$$

and g_s is the electro-weak renormalized coupling or strong quark gluon coupling [15].

For $M = 0$ (gluon renormalized mass, see equation (33)) we approximate the exact $A(p^2)$ in a simple way by adding an infinite number of Feynman graphs that contain only one gluon at any time:

$$\begin{aligned} A(p^2) \;&\simeq\; \bar{A}(p^2) = Y\left\{\frac{1}{D} + \frac{1}{D}C\frac{1}{D} + \frac{1}{D}C\frac{1}{D}C\frac{1}{D}\dots\right\}X = Y\frac{1}{D-C}X && (34)\\ D \;&=\; \gamma_\mu p^\mu - m\sqrt{1 + f(p^2/m^2)} && (35)\end{aligned}$$

All other contributions that we neglect here contain two or more virtual gluons simultaneously in the intermediate states: they are supposed to be less relevant starting with terms of order g^4. Hence from equation (34) we may obtain:

$$\tilde{A}(p^2) = Y\frac{1}{\gamma\cdot p - m\sqrt{1+f} - C}X = Y\frac{1}{\gamma\cdot p - m\sqrt{1+f} - \tilde{A}\not{p} - \tilde{B}}X \tag{36}$$

The search of poles of $\tilde{A}(p^2) \simeq A(p^2)$ leads to the equation

$$p^2(1-\tilde{A})^2 - m^2\left[\sqrt{1+f(x)} + \frac{\tilde{B}(p^2)}{m}\right]^2 = 0 \tag{37}$$

Under the approximation $\tilde{A}(p^2) \simeq \tilde{A}(m^2)$ and $\tilde{B}(p^2) \simeq \tilde{B}(m^2)$ that follow from the very reasonable assumption $\tilde{A} \ll 1,\ \tilde{B} \ll m\sqrt{1 + f(p^2/m^2)}$ we obtain the equation

$$x = \frac{p^2}{m^2} = \left(\frac{\sqrt{1+f(x)} + \frac{\tilde{B}(m^2)}{m}}{1 - \tilde{A}(m^2)}\right)^2 \tag{38}$$

which for $g_s \to 0,\ \tilde{A}, \tilde{B} \to 0$ gives back the classical equation

$$\frac{p^2}{m^2} = 1 + f\left(\frac{p^2}{m^2}\right) \qquad\qquad x = 1 + f(x)\,, \qquad x = \frac{p^2}{m^2} \tag{39}$$

Note that equation (39) provides poles of p^2 at zero order (no coupling). More specifically, in order to make the integral (33) finite we can take

$$f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 \tag{40}$$

so that to solve (39) will now mean to find the three zeros x_0, x_+ and x_- of $x - f(x) - 1$. We can then write

$$x - f(x) - 1 = -\lambda_3(x - x_0)(x - x_+)(x - x_-)$$

and since we know that $f(1) = 0$, on the one hand we have $\lambda_0 = -\lambda_1 - \lambda_2 - \lambda_3$, and on the other we can always put $x_0 = 1$. As a consequence it is easy to see that

$$\frac{\lambda_2}{\lambda_3} = -1 - x_+ - x_+ \qquad \frac{\lambda_1 - 1}{\lambda_3} = x_+ + x_- + x_+x_- \qquad x_\pm = \frac{-\lambda_2 - \lambda_3 \pm \sqrt{\Delta}}{2\lambda_3}$$
$$\Delta = (\lambda_2 - \lambda_3)^2 - 4\lambda_1\lambda_3 - 4\lambda_3^2 + 4\lambda_3$$

Then, if following our model the three masses are $m_1 = m$, $m_2 = m\sqrt{x_+}$ and $m_3 = m\sqrt{x_-}$, we finally have

$$x_0 = 1 \qquad x_+ = \frac{m_3^2}{m_1^2} \qquad x_- = \frac{m_2^2}{m_1^2} \tag{41}$$

We now remind that the field theories we are considering here (QED, QCD and $SU_C(3) \times SU_L(2) \times U(1)$) are renormalizable and consequently the function $f(x)$ can be looked at as a smooth cut-off that regularizes the perturbative terms. *More precisely the integral C becomes finite (integrand convergent) under the assumption that the function $f(x)$ is analytic and appears either as a series expansion or as a polynomial of third degree at least.* In the latter case one obtains three poles in the (zero order) fermionic propagator and obviously three physical masses for appropriate values of its coefficients. *We may relate this result with the three families of fundamental particles (quarks) thus describing the flavour phenomenon*: more explicitly two propagators (charge $-\frac{1}{3}$ and $+\frac{2}{3}$) for quarks, one for charged leptons (e^-, μ^-, τ^-) and one for neutrinos (ν_e, ν_μ, ν_τ). However the numerical values of these masses cannot be considered valid if compared with with experimental data, except for particular cases where the strong and e.m. interactions are absent (see neutrinos in the next section).

We now consider the Feymnam propagator for fundamental bosons (gluons, $W^\pm$, Z, Higgs). Again we develop it in perturbation theory and discover that the most significant contribution that must be added to the zero order derives from the fermion-antifermion loops. To be precise one has the zero order propagator that connects a fermion vertex γ_μ with another γ_ν: $A = A^{(0)} + A^{(1)} + \dots$

$$A^{(0)} = \frac{1}{p^2 - M^2}\left[g_{\mu\nu} + (\xi^2 - 1)\frac{p_\mu p_\nu}{p^2 - \xi^2 M^2}\right] \tag{42}$$

where $\xi = 1$ in t'Hooft-Feynman gauge, $\xi = \infty$ in unitary gauge and $\xi = 0$ in Landau gauge

$$A^{(1)} = 4\pi g^2 \int \frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{1}{q - p - m\sqrt{1 + f\left(\frac{(q-p)^2}{m^2}\right)}}\gamma_\nu \frac{1}{q - m\sqrt{1 + f\left(\frac{q^2}{m^2}\right)}}\gamma_\nu\right\} \tag{43}$$

IV. NEUTRINOS

Within the scenario of weak interactions, neutrinos need a special attention. Indeed they are considered to be massless in the Standard Model but experiments have shown that they have masses. These masses are much smaller than those of other fundamental particles. Neutrinos are produced by weak interactions in definite flavor states (ν_e, ν_μ, ν_τ) which are not stationary mass (energy) states and one expects mixing and oscillations between the various flavor states. Both mixing and oscillations are confirmed by experiments. The oscillations are permanent because neutrinos do not decay [16, 17].

According to our approach the propagator of a particle of spin $\frac{1}{2}$ with mass is given by formula (29) and in the neutrino case we may deal with an $f(x)$ as a third order polynomial thus obtaining three physical masses $m_1 < m_2 < m_3$, in both cases of Dirac or Majorana neutrinos[3]. The Lagrangian of interaction neutrinos/other particles can be divided into two parts $\mathcal{L}_I^{CC}$ (charged current) $\mathcal{L}_I^{NC}$ (neutral current):

$$\mathcal{L}_I^{CC} = -\frac{g}{2\sqrt{2}}\,J_\alpha^{CC} W^\alpha \qquad \mathcal{L}_I^{NC} = -\frac{g}{2\cos\theta_W}\,J_\alpha^{NC} Z^\alpha \tag{44}$$

[3] This alternative has not been established yet on experimental grounds.

g electroweak coupling, θ_W weak angle, W^α, Z^α are $W^\pm, Z^0$, and $J_\alpha^{CC}, J_\alpha^{NC}$ charged and neutral currents respectively. With the neutrino masses tending to zero the interactions conserve L_e, L_μ, L_τ lepton numbers:

$$\sum L_e = const \qquad \sum L_\mu = const \qquad \sum L_\tau = const \tag{45}$$

with non vanishing masses the neutrino mass term in field theory does not conserve lepton numbers. We have

$$\nu_{\ell L} = \sum_i U_{\ell i}\nu_i \tag{46}$$

where ν_i is the field of mass m_i and U is the unitary mixing matrix.

For neutrinos with definite masses there are two possibilities: if the total $L = L_e + L_\mu + L_\tau$ is conserved they are Dirac particles (four component spinors), if there are not any conserved lepton numbers they are two component Majorana particles[4] (no difference between neutrinos and antineutrinos). Neutrino-less double β-decay $(A, Z) \to (A, Z + Z) + e^- + e^-$ is forbidden if massive neutrinos are Dirac particles.

If neutrinos are Dirac particles the mixing parameters are three rotation angles $\theta_{12}, \theta_{23}, \theta_{13}$ plus a phase factor δ; if they are Majorana's there are two more phases α and β that are irrelevant for the oscillations and play a role only in the neutrino-less, double-β decay [16]. More specifically we have

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \cdot \begin{pmatrix} C_{13} & 0 & S_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -S_{13}e^{i\delta} & 0 & C_{23} \end{pmatrix} \cdot \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \tag{47}$$

where $S_{ij} = \sin\theta_{ij}$ and $C_{ij} = \cos\theta_{ij}$. According to the previous arguments one easily obtains the probabilities, for neutrinos, of changing or not changing flavors in vacuum (see the case of atmospheric neutrinos).

If we reduce our analysis to the transition $\nu_\mu \rightleftarrows \nu_\tau$ in vacuum we obtain

$$\boldsymbol{P}(\nu_\mu \to \nu_\tau, L) = |\langle \nu_\tau, L|\nu_\mu, 0\rangle|^2 = C_{23}^2 S_{23}^2 \sin^2 \frac{p_3 - p_2}{2} L \simeq C_{23}^2 S_{23}^2 \sin^2 \frac{\Delta m_{23}^2}{4E} \tag{48}$$

with $\Delta m_{23}^2 = m_3^2 - m_2^2$, and $\boldsymbol{P}(\nu_\mu \to \nu_\mu, L) = 1 - \boldsymbol{P}(\nu_\mu \to \nu_\tau, L)$, L being the distance at which we detect ν.

While the oscillations of neutrinos in the vacuum are a kinematical phenomenon, in highly dense media it becomes a dynamical phenomenon because of the interaction neutrino-matter[5] (electrons, quarks ...) mainly due to ν_e-e interaction. The nuclear fusion reactions in the core of the Sun produce electron neutrinos in a high density medium. Afterwards neutrinos cross a decreasing density medium before reaching the surface of the Sun [16].

One then can describe this situation with a phenomenological potential $V(r) = \sqrt{2}G_F N_e(r)$, G_F is the Fermi constant, $N_e(r)$ is the electron density at a distance r from the centre of the Sun. Accordingly we must deal with neutrino effective masses and effective mixing angles; they are different from those in the vacuum. In particular one obtains for the angle θ_{12}^m the following formula[6]

$$\tan 2\theta_{12}^m = \frac{\delta_m^2 \sin 2\theta_{12}}{\delta_m^2 \cos 2\theta_{12} - A} \tag{49}$$

with $\delta_m^2 = m_2^2 - m_1^2$, $A = 2\sqrt{2}G_F N_e E$, and E is the neutrino energy [16, 17].

Numerical values: The function $f(x)$ introduced in this and in the previous note, in the approximated form of a third-degree polynomial, permits to calculate masses and other physical properties of the fundamental particles, namely quarks, neutrinos, gluons, charged leptons ... The calculation is particularly simple, qualitatively and quantitatively significant, for the massive neutrinos (m_1, m_2, m_3) from the data [17] (whose values might change) where

[4] Processes in which the total L is conserved like $\mu \to e + \gamma$ or similar ones are permitted in case of mixing Dirac massive neutrinos (probabilities much smaller than the experimental upper bounds).

[5] All neutrinos interact with electrons and quarks by neutral currents, but only electronic neutrinos interact with electrons and quarks via charged currents.

[6] Notice the resonance condition ($\theta_{12}^m = \pi/4$) at the density

$$N_e = \frac{\delta_m^2 \cos 2\theta_{12}}{E2\sqrt{2}G_F}$$

one can ignore the electro-weak coupling and deal with the zero order only:

$$\delta_m^2 = m_2^2 - m_1^2 \simeq 76.6 \pm 3.5\,(meV)^2 \tag{50}$$

$$|\Delta m^2| = m_3^2 - \frac{1}{2}(m_2^2 - m_1^2) \simeq 2\,380 \pm 270\,(meV)^2 \tag{51}$$

Indeed we must consider the function $f(x)$ defined in (40) as a universal function for all weak interactions. Consequently the zeros of $x - f(x) - 1$ should almost coincide with those of the charged leptons and for the neutrino masses m_1, m_2, m_3. More precisely, if we assume on the basis of universality that the ratios of the masses w.r.t. the lighter one are identical in both cases, we obtain:

$$x_0 = 1 = \frac{m_1^2}{m_1^2} = \frac{m_e^2}{m_e^2} \qquad x_- = \frac{m_\mu^2}{m_e^2} = \frac{m_2^2}{m_1^2} \qquad x_+ = \frac{m_\tau^2}{m_e^2} = \frac{m_3^2}{m_1^2} \tag{52}$$

$$x_- m_1^2 - m_1^2 \simeq 76.6\,(meV)^2 \qquad m_1 \simeq 4.235 \times 10^{-5}\,eV \qquad m_2 \simeq 6 \times 10^{-3}\,eV \qquad m_3 \simeq 1.4 \times 10^{-1}\,eV \tag{53}$$

From these numbers we get

$$|\Delta m^2| \simeq 2.15 \times 10^{-2}\,(eV)^2 \tag{54}$$

In an analogous way on may start from assuming $\Delta m^2 \simeq 2.38 \times 10^3\,(meV)^2$ on the basis of the analysis of the data [17], and obtain the following values

$$m_1 \simeq 1.42 \times 10^{-5}\,eV \qquad m_2 \simeq 2.928 \times 10^{-3}\,eV \qquad m_3 \simeq 4.924 \times 10^{-1}\,eV \tag{55}$$

thus obtaining $\delta_m^2 \simeq 8.57 \times 10^{-4}\,(eV)^2$. A similar calculation becomes unrealistic for quarks, and basic bosons because of the strong coupling that is present in addition to weak and electromagnetic forces.

V. CONCLUSIONS

As we proposed, the insertion of an analytic function $f(x)$ (for $x = p^2/m^2$) into the relativistic energy-momentum formula of the forceless particle preserves the relation between a certain Lévy stochastic process and quantum relativistic mechanics as in the usual case [1, 2]. Furthermore the phenomenological reduction of $f(x)$ as a third degree polynomial makes the Feynman graphs convergent at any order (due to renormalizable theories) and creates three poles (in the x-variable)that may be related qualitatively with three masses, and consequently with the flavor phenomenon (see details in the Section III). Hence we derive two propagators for quarks (charge $-\frac{1}{3}$ and $+\frac{2}{3}$), one for charged leptons and one for neutrinos.

As far as neutrinos are concerned we obtain quantitative results that are almost compatible with the experimental data [17]. This is possible (in neutrino's case) because they couple among themselves or with other leptons via weak interactions only and $f(x)$ can be assumed universal within the weak interaction scenario. Hence we can calculate the neutrino mass spectrum from the zero order propagator, and ignore the expansion of the renormalized weak coupling. Impossible to consider only the zero order propagator for other fundamental particles (such as quarks etc...) because of their e.m. and strong couplings.

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