Mass spectrum and Lévy–Schrödinger relativistic equation

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We introduce a modification in the relativistic equations in such a way that (1) the relativistic Schrödinger equations can always be based on an underlying Lévy process, (2) several families of particles with different rest masses can be selected, and finally (3) the corresponding Feynman diagrams are convergent when we have at least three different masses.

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I. INTRODUCTION AND NOTATIONS

In this note we adopt the space-time relativistic approach of Feynman's propagators (for bosons and fermions) instead of the canonical Lagrangian-Hamiltonian quantized field theory. Indeed the former alternative is preferred to the latter for the developments of our basic ideas that exhibit the connection between the propagator of quantum mechanics and Lévy's stochasticity. More precisely the relativistic Feynman propagators are here linked with a dynamical theory based on a particular Lévy stochastic process. This point, already mentioned in a previous paper [1], is here analyzed thoroughly with the purpose of deducing its consequences for the case of fundamental fermions and bosons (quarks, leptons, gluons etc...) of the Standard Model (SM) characterized by the symmetry $SU_C(3) \times SU_L(2) \times U(1)$. To this end we now recall that a Lévy process is a stochastic process $X(t)$, $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- $X(0) = 0, \quad \mathbb{P}$-q.o.
- $X(t)$ has independent and stationary increments: for each n and for very choice of $0 \leq t_1 < t_2 < \ldots < t_n < +\infty$ the increments $X(t_{j+1}) - X(t_j)$ are independent and $X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - X(t_j)$;
- $X(t)$ is stochastically continuous: for every $a > 0$ and for every s

$$\lim_{t \to s} \mathbb{P}\left(|X(t) - X(s)| > a\right) = 0.$$

To simplify the notation we will consider in the following one–dimensional (the n–dimensional extension would not be a difficult task) Lévy processes: it is well known [2–4] that all its laws are infinitely divisible, but we will be mainly interested in the non stable (and in particular non Gaussian) case. In other words the characteristic function of the process Δt–increment is $[\varphi(u)]^{\Delta t/\tau}$ where φ is infinitely divisible, but not stable [1], and τ is a time scale parameter. The transition probability density $p(2|1) = p(x_2, t_2|x_1, t_1)$ of a particle moving from the space-time point 1 to 2 then is

$$p(2|1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du\, [\varphi(u)]^{(t_2 - t_1)/\tau} e^{-iu(x_2 - x_1)} \tag{1}$$

In analogy with the non relativistic Wiener case we obtain for the motion of a free particle the Feynman propagator $\mathcal{K}(2|1) = \mathcal{K}(x_2, t_2|x_1, t_1)$ as

$$\mathcal{K}(2|1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du\, [\varphi(u)]^{i(t_2 - t_1)/\tau} e^{-iu(x_2 - x_1)} \tag{2}$$

and the corresponding wave function evolution is

$$\psi(x, t) = \int_{-\infty}^{+\infty} dx'\, \mathcal{K}(x, t|x', t')\psi(x', t'). \tag{3}$$

From (2) and (3) we easily obtain [1]

$$i\partial_t \psi = -\frac{1}{\tau}\eta(\partial_x)\psi$$

where $\eta = \log \varphi$ and $\eta(\partial_x)$ is a pseudodifferential operator with symbol $\eta(u)$ defined through the use of Fourier transforms [3, 5–7]. It plays the role of the generator of

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[1] A law φ is said to be infinitely divisible if for every n it exists a characteristic function φ_n such that $\varphi = \varphi_n^n$; on the other hand it is said to be stable when for every $c > 0$ it is always possible to find $a > 0$ and $b \in \mathbf{R}$ such that $e^{ibu}\varphi(au) = [\varphi(u)]^c$. Every stable law is also infinitely divisible.

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the semigroup $T_t = e^{t\eta(\partial_x)/\tau}$ operating on the Banach space of the measurable, bounded functions [3, 5–7].

It is very well known [2, 3], on the other hand, that φ represents an infinitely divisible law if and only if $\eta(u) = \log\varphi(u)$ satisfies the Lévy–Khintchin formula

$$\eta(u) = i\gamma u - \frac{\beta^2 u^2}{2} + \int_{\mathbb{R}} \left[e^{iux} - 1 - iux\, I_{[-1,1]}(x)\right]\, \nu(dx) \tag{4}$$

where $\gamma, \beta \in \mathbb{R}$, $I_{[-1,1]}(x)$ is the indicator of $[-1, 1]$, and $\nu(dx)$ is the Lévy measure, namely a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(dx) < +\infty.$$

In the case of a centered, symmetric law the equation (4) simplifies in

$$\eta(u) = -\frac{\beta^2 u^2}{2} + \int_{\mathbb{R}} (\cos ux - 1)\, \nu(dx) \tag{5}$$

and $\eta(u)$ becomes even and real. As a consequence the corresponding operator $\eta(\partial_x)$ is self–adjoint and acts on propagators and wave functions according to the Lévy–Schrödinger integro–differential equation

$$\begin{aligned} i\partial_t\psi(x,t) &= -\frac{1}{\tau}\,\eta(\partial_x)\psi(x,t) \\ &= -\frac{\beta^2}{2\tau}\,\partial_x^2\psi(x,t) \\ &\quad -\frac{1}{\tau}\int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)]\;\nu(dy). \end{aligned} \tag{6}$$

The integral term accounts for the jumps in the trajectories of the underlying stochastic process, while an action α with $\beta^2 = \alpha\tau/m$ provides the usual differential term of the Schrödinger equation. For $\beta = 0$ a pure jump Lévy–Schrödinger equation is obtained

$$i\partial_t\psi(x,t) = -\frac{1}{\tau}\int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)]\;\nu(dy). \tag{7}$$

II. STATIONARY SOLUTIONS FOR THE FREE PARTICLE

Equation (6) allows a simple stationary solution: if we consider

$$\psi(x,t) = e^{-iE_0 t/\alpha}\phi(x), \qquad \alpha = \frac{m\beta^2}{\tau}$$

we have then

$$E_0\phi(x) = -\frac{\alpha^2}{2m}\,\phi''(x) - \frac{\alpha}{\tau}\int_{\mathbb{R}} [\phi(x+y) - \phi(x)]\,\nu(dy), \tag{8}$$

and for a plane wave $\phi(x) = e^{\pm iux}$ from (5) with a symmetric Lévy noise

$$\begin{aligned} E_0\phi(x) &= -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}} \left(e^{\pm iuy} - 1\right)\,\nu(dy)\right] e^{\pm iux} \\ &= -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}} (\cos uy - 1)\,\nu(dy)\right]\phi(x) \\ &= -\frac{\alpha}{\tau}\eta(u)\phi(x) \end{aligned}$$

which is satisfied when $E_0 = -\alpha\eta(u)/\tau$. Finally by taking $p = \alpha u$ for the momentum we obtain the relevant equation

$$E_0 = -\frac{\alpha}{\tau}\,\eta\left(\frac{p}{\alpha}\right). \tag{9}$$

III. RELATIVISTIC QUANTUM MECHANICS

Equation (9) connects the kinetic energy of a forceless particle to the logarithmic characteristic of a Lévy process: if we take in particular the non stable law $\eta(u) = 1 - \sqrt{1 + a^2u^2}$ with the following identification of the parameters

$$\alpha = \hbar, \qquad \frac{\hbar}{\tau} = mc^2, \qquad a = \frac{\hbar}{mc}, \qquad p = \hbar u.$$

we are led to the formula

$$E_0 = -mc^2\eta\left(\frac{p}{\hbar}\right) = E - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2 \tag{10}$$

which is the well–known relativistic kinetic energy for a particle of mass m. The Schrödinger equation of a relativistic free–particle is easily obtained from (10) by reinterpreting as usual E and p respectively as the operators $i\hbar\partial_t$ and $-i\hbar\partial_x$:

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - \hbar^2c^2\partial_x^2}\,\psi(x,t), \tag{11}$$

but this comes also from (6) after having absorbed the mass energy term $-mc^2$ of (10) into a phase factor $e^{imc^2t/\hbar}$. In three dimensions (11) reads

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - \hbar^2c^2\boldsymbol{\nabla}^2}\,\psi(x,t) \tag{12}$$

It has been shown [3, 8] that the Lévy process behind the equations (11) and (12) is a pure jump process [1, 3] with an absolutely continuous Lévy measure $\nu(dx) = W(x)dx$ and

$$W(x) = \frac{1}{\pi|x|}\,K_1\left(\frac{|x|}{a}\right) = \frac{1}{\pi|x|}\,K_1\left(\frac{mc}{\hbar}\,|x|\right) \tag{13}$$

(K_ν are the modified Bessel functions [9]), that in three dimensions becomes

$$W(\boldsymbol{x}) = \frac{1}{2a\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{|\boldsymbol{x}|}{a}\right) = \frac{mc}{2\hbar\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{x}|\right) \tag{14}$$

while from (7) the equation (11) becomes equivalent to

$$i\hbar\partial_t\psi(x,t) \qquad (15)$$
$$= -mc^2\int_{\mathbb{R}}\frac{\psi(x+y,t)-\psi(x,t)}{\pi|y|}\,K_1\left(\frac{mc}{\hbar}\,|y|\right)dy$$

and in three dimensions

$$i\hbar\partial_t\psi(\boldsymbol{x},t) \qquad (16)$$
$$= -mc^2\!\int_{\mathbb{R}^3}\frac{\psi(\boldsymbol{x}+\boldsymbol{y},t)-\psi(\boldsymbol{x},t)}{2\pi^2|\boldsymbol{y}|^2}\,\frac{mc}{\hbar}K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{y}|\right)d^3\boldsymbol{y}$$

From the equation (12) by the well known standard procedures [10] one derives (always for the free particle) the Klein–Gordon and Dirac equations in three dimensions for the wave functions and spinors ψ, respectively

$$\left(\Box-\frac{m^2c^2}{\hbar^2}\right)\psi \;=\; 0, \qquad (17)$$
$$\left(i\gamma_\mu\partial^\mu-\frac{mc}{\hbar}\right)\psi \;=\; 0. \qquad (18)$$

The Klein–Gordon and Dirac propagators verify instead the inhomogeneous equations (here $\hbar=c=1$)

$$\left(\Box_2-m^2\right)\mathcal{K}_{KG}(2|1) \;=\; \delta^4(2|1) \qquad (19)$$
$$\left(i\gamma_\mu\partial_2^\mu-m\right)\mathcal{K}_D(2|1) \;=\; i\,\delta^4(2|1) \qquad (20)$$

with $\delta^4(2|1)=\delta(t_2-t_1)\delta^3(\boldsymbol{x}_2-\boldsymbol{x}_1)$. Let us finally remark that these relativistic quantum wave equations have been recently of particular interest [11] also in the field of quantum optical phenomena and of quantum information.

IV. INFINITE DIVISIBILITY–PRESERVING MODIFICATIONS

The relativistic, time–like four–momentum $p=(E/c\,,\boldsymbol{p})$ obeys the relation (here $\boldsymbol{p}^2$ will represent the square modulus of the tri–vector $\boldsymbol{p}$)

$$p^2=\frac{E^2}{c^2}-\boldsymbol{p}^2=m^2c^2\geq 0 \qquad (21)$$

so that the hamiltonian dependence of energy on momentum is

$$E(\boldsymbol{p})=\sqrt{m^2c^4+\boldsymbol{p}^2c^2}=mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}\,. \qquad (22)$$

This relativistic energy E containing a rest mass term mc^2, the kinetic energy E_0 in a dimensionless form becomes

$$\frac{E_0(\boldsymbol{p})}{mc^2}=\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}-1. \qquad (23)$$

By taking now

$$\eta=-\frac{E_0}{mc^2}, \qquad \boldsymbol{u}=\frac{\boldsymbol{p}}{amc}$$

where a is a constant with the dimensions of a length, η will be dimensionless while $\boldsymbol{u}$ will be the reciprocal of a length, and the equation (23) becomes

$$\eta(\boldsymbol{u})=1-\sqrt{1+a^2\boldsymbol{u}^2}$$

namely – not surprisingly – it takes the three-dimensional form of the logarithmic characteristic giving rise to the relativistic quantum equations in the Section III.

Our purpose consists now in proposing a modification of $\eta(\boldsymbol{u})$ that preserves the infinite divisibility of the law, and eventually produces changes in the forceless equations of motion for a particle – both at the classical and at the quantum level – in comparison with the classical and quantum motions given by the equations (17) and (18). To this end we modify (22) in the following way

$$E(\boldsymbol{p})=mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}+f\left(\frac{p^2}{m^2c^2}\right)} \qquad (24)$$

where f is a – possibly small – dimensionless, smooth function of the relativistic scalar p^2/m^2c^2. Of course this modification entails that p^2 no longer coincides with m^2c^2 since the standard relation (21) is now changed into

$$p^2=\frac{E^2}{c^2}-\boldsymbol{p}^2=m^2c^2+m^2c^2f\left(\frac{p^2}{m^2c^2}\right). \qquad (25)$$

As we will see in the following this also implies that the mass no longer is m: it will take instead one or more values depending on the choice of f. In fact it could appear to be preposterous to introduce a function f of an argument which after all is a constant (albeit different from 1). However we will show that this will lend us the possibility of having both a mass spectrum, and a new wave equation when – in the next section – we will quantize our classical relations. Moreover it will be argued in the following that in this way the corresponding modified logarithmic characteristic η will remain infinitely divisible: a feature that is instrumental for a connection to a suitable underlying Lévy process.

To see that we first remark that (25) defines the total particle energy E in an implicit form. To find it explicitly we rewrite (25) in a dimensionless form as

$$\frac{p^2}{m^2c^2}=1+f\left(\frac{p^2}{m^2c^2}\right),$$

and then, by taking $g(x)=x-f(x)$, we just observe that the former equation requires that x be solution of $g(x)=1$, namely

$$g\left(\frac{p^2}{m^2c^2}\right)=\frac{p^2}{m^2c^2}-f\left(\frac{p^2}{m^2c^2}\right)=1.$$

If then $g^{-1}(1)$ represents one of the (possibly many) solutions of this equation, we could write

$$\frac{p^2}{m^2c^2}=g^{-1}(1)$$

so that we have

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2g^{-1}(1)$$

which can be interpreted as a simple mass re-scaling from m to one of the (possibly many) values $M = m\sqrt{g^{-1}(1)}$. The new hamiltonian then is

$$E(\boldsymbol{p}) = \sqrt{m^2c^4g^{-1}(1) + \boldsymbol{p}^2c^2} = Mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{M^2c^2}} \quad (26)$$

and its kinetic part (by applying the same re-scaling also to the subtracted rest mass term) is

$$\begin{aligned} E_0(\boldsymbol{p}) &= E(\boldsymbol{p}) - mc^2\sqrt{g^{-1}(1)} \\ &= Mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{M^2c^2}} - Mc^2. \end{aligned}$$

Hence the main consequence of our modification consists of a re-scaling of the mass value ($m \to M$) at a purely classical level. This fact is apparently helpful because it is straightforward to see now that the new associated logarithmic characteristic η is again infinitely divisible, and hence still produces acceptable Lévy processes. But there is more: since $g^{-1}(1)$ can take several different real and positive values, by means of our modification (24) we have introduced an entire mass spectrum: indeed in the rest frame of the particle we have

$$M = E_{cm}/c^2 = m\sqrt{g^{-1}(1)} \quad (27)$$

V. QUANTUM EQUATIONS OF MOTION

It is important to remark that while the equation (24) allows a peculiar transition to quantum mechanics ($E \to i\hbar\partial_t$, $\boldsymbol{p} \to -i\hbar\boldsymbol{\nabla}$) if we interpret this energy formula as a new hamiltonian operator, namely it leads to

$$i\hbar\partial_t\psi(x,t) = mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\boldsymbol{\nabla}^2 + f\left(\frac{\Box}{m^2c^2}\right)}\,\psi(x,t) \quad (28)$$

the equation (26) gives instead the usual Klein–Gordon equation (12) with just a possibly re-scaled mass $M = m\sqrt{g^{-1}(1)}$. In fact from (28) one obtains now a *modified* Klein–Gordon equation for both the wave function ψ and its corresponding propagator $\mathcal{K}_{KG}(2|1)$ (from here on $\hbar = c = 1$)

$$\left[\Box - m^2 f\left(\frac{1}{m^2}\Box\right) - m^2\right]\psi = 0, \quad (29)$$

$$\left[\Box_2 - m^2 f\left(\frac{1}{m^2}\Box_2\right) - m^2\right]\mathcal{K}_{KG}(2|1) = \delta^4(2|1) \quad (30)$$

and by standard methods [10] the *modified* Dirac spinor equations

$$\left[i\gamma_\mu\partial^\mu - m\sqrt{1 + f\left(\frac{1}{m^2}\Box\right)}\right]\psi = 0 \quad (31)$$

$$\left[i\gamma_\mu\partial_2^\mu - m\sqrt{1 + f\left(\frac{1}{m^2}\Box_2\right)}\right]\mathcal{K}_D(2|1) = i\delta^4(2|1) \quad (32)$$

In the momentum space (with Fourier transforms in four dimensions) these equations become much simpler: more precisely we have

$$\begin{aligned} \mathcal{K}_{KG}(p^2) &= \frac{1}{p^2 - m^2\left[1 + f(p^2/m^2)\right] + i\epsilon} \\ \mathcal{K}_D(p^2) &= \frac{1}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} \end{aligned}$$

We notice that $\mathcal{K}_D(2|1)$ is in our case simply related to the $\mathcal{K}_{KG}(2|1)$ (like in the usual case) as

$$\mathcal{K}_D(2|1) = i\left(i\not\partial_2 + m\sqrt{1 + f(\Box_2/m^2)}\right)\mathcal{K}_{KG}(2|1)$$

VI. PHENOMENOLOGY: QUARK AND LEPTON MASSES

The equations (30) and (32) generalize the well known propagator equations (19) and (20) which derive from QED and QCD at zero order (in absence of interaction terms). For future developments we recall that the Lagrangian density of QCD is[2], up to gauge fixing terms:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \sum_q \overline{\psi}_i^q[i\gamma^\mu(D_\mu)_{ij} - m_q\delta_{ij}]\psi_j^q$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc}A^b_\mu A^c_\nu$, and the insertion of interaction terms is done with the minimal interaction by substituting the simple derivative ∂_μ with the covariant one D_μ where we have respectively for QED and QCD

$$\begin{aligned} D_\mu &\equiv \partial_\mu - ieA_\mu \\ (D_\mu)_{ij} &\equiv \delta_{ij}\partial_\mu - ig_sT^a_{ij}A^a_\mu. \end{aligned}$$

The Standard Model (SM) $SU_c(3)\times SU_L(2)\times U(1)$ treats both strong and electro–weak interactions: within this scheme the modified $\eta(\boldsymbol{u})$ leads to new interesting consequences. We begin by considering the Feynman rules in

[2] Here g_s is the QCD coupling constant, T^a_{ij} and f_{abc} are the $SU(3)$ color matrices and structure constants respectively, and A^a_μ the eight Yang–Mills gluon fields; ψ^q_i are the Dirac 4-spinors associated with each quark field of color i and flavor q.

perturbation theory in presence of the modified zero order propagator for both spin $\frac{1}{2}$ (quarks and leptons) and spin 1 (gluons, vector weak interacting Bosons). The amplitude A for a fermion that propagates from vertex X to vertex Y if expanded looks as follows: $A = A^{(0)} + A^{(1)} + A^{(2)} + \dots$ The lowest order is

$$A^{(0)} = Y \frac{i}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} X.$$

It is then possible that the Fermion emits and reabsorbs a virtual vector boson from X to Y:

$$\begin{aligned} A^{(1)} &= 4\pi g_s^2 Y \int d^4k \, \frac{\gamma^\mu}{\gamma^\rho p_\rho - m\sqrt{1+f(p^2/m^2)}} \, \frac{1}{(p-k)^2} \\ &\times \frac{1}{k^\nu \gamma_\nu - m\sqrt{1+f(k^2/m^2)} + i\epsilon} \\ &\times \frac{\gamma_\mu}{\gamma^\rho p_\rho - m\sqrt{1+f(p^2/m^2)}} X \end{aligned}$$

We choose now $f(x)$ in such a way that it makes finite the integral

$$C = \gamma^\mu \int \frac{d^4k}{\gamma^\rho k_\rho - m\sqrt{1 + f\,(k^2/m^2)} + i\epsilon} \, \frac{1}{(p-k)^2} \gamma_\mu \tag{33}$$

One may notice that $f(x)$ behaves as a smooth *cut-off* in a procedure of Regularization at each order in QCD (and QED). The integral C is an invariant of the form $C = A(p^2)\not{p} - B(p^2)$ and its integrand is also present as a factor in higher order terms, thus producing convergence. In a similar way one expects that the representation of the complete fermionic propagator, as well as its zero-order, is made up of two additive terms, in momentum space, each of them exhibiting simple analyticity properties except for a limited number of poles and branch points[3] [14]: for the zero order we have

$$\frac{i}{\not{p} - m\sqrt{1 + f(p^2/m^2)}} = i\,\frac{\not{p} + m\sqrt{1+f(p^2/m^2)}}{p^2 - m^2[1 + f(p^2/m^2)]}$$

Let us now reconsider the equation (33). The simplest expression for $f(x)$ compatible with C finite is a polynomial of third degree in x:

$$f(x) = \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 \tag{34}$$

and this is suggestively connected with the possibility of having a mass spectrum. Indeed, as stated in the Section IV, the spectrum is produced by the multiple solutions of the equation $g(x) = x - f(x) = 1$, and we achieve three values that, under proper conditions, might correspond to three masses. If we consider the

[3] Likewise one can reason for the complete and zero order propagator of basic bosons (gluons, $W^\pm, Z^0$, Higgs).

m_d	m_s	m_b
3×10^{-3}	70×10^{-3}	4.13
7×10^{-3}	120×10^{-3}	4.27
λ_1	λ_2	λ_3
-1.84×10^{-3}	1.84×10^{-3}	-9.69×10^{-10}
-3.41×10^{-3}	3.41×10^{-3}	-9.14×10^{-9}

TABLE I: Estimated values of the λ's in (34) for quarks with charge $-\frac{1}{3}$. Masses are in Gev/c^2.

m_u	m_c	m_t
1.5×10^{-3}	1.16	171.2
3.0×10^{-3}	1.34	174.0
λ_1	λ_2	λ_3
-1.67×10^{-6}	1.67×10^{-6}	-1.28×10^{-16}
-5.01×10^{-6}	5.01×10^{-6}	-1.49×10^{-15}

TABLE II: Estimated values of the λ's in (34) for quarks with charge $\frac{2}{3}$. Masses are in Gev/c^2.

m_e	m_μ	m_τ
5.11×10^{-4}	105.6×10^{-3}	1.77
λ_1	λ_2	λ_3
-2.35×10^{-5}	2.35×10^{-5}	-1.95×10^{-12}

TABLE III: Estimated values of the λ's in (34) for leptons with charge -1. Masses are in Gev/c^2.

three (real and positive) zeros x_1, x_2 and x_3 of the polynomial $g(x) - 1 = x - f(x) - 1$ we easily find the following simple algebraic relations with the λ's:

$$\begin{aligned} \lambda_1 &= 1 - \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \right) \\ \lambda_2 &= \frac{1}{x_1 x_2} + \frac{1}{x_1 x_3} + \frac{1}{x_2 x_3}\,, \qquad \lambda_3 = \frac{-1}{x_1 x_2 x_3} \end{aligned}$$

The connections with the possible experimental physical masses are $M_1 = m\sqrt{x_1}$, $M_2 = m\sqrt{x_2}$, $M_3 = m\sqrt{x_3}$. If the three poles in the free (zero order) propagator are real and positive (with proper residues), with appropriate values of the λ's, they allow the interpretation of physical basic masses of fermions (quark or leptons) belonging to the three different families of the Standard Model. To be more specific we get two different propagators for quarks, one with charge $-\frac{1}{3}$ (d, s, b quarks) and another with charge $+\frac{2}{3}$ (u, c, t quarks). Similarly for charged leptons (charge -1 and spin $\frac{1}{2}$) we get one propagator.

In the Tables I, II and III we give a few examples of numerical values of the λ parameters in equation (34) for quark and lepton masses (in Gev/c^2) taken from the Particle Data Group [13]. For the quarks with charge $-\frac{1}{3}$ and different mass estimates we obtain the results listed in the Table I, while for quarks with charge $\frac{2}{3}$ we get the results of the Table II, and finally for charged leptons with charge -1 we have the results of Table III. According to our model a significant contribution to the fermion mass spectrum derives from the poles of the zero order propagator, whereas the role of the interaction terms might become complementary and can be estimated in (renormalized) perturbation theory.

Our λ's also produce the regularization at each order in QCD (and QED) perturbation theory. Furthermore we know that after regularization the approximate representation of the propagator tends to a finite limit (exact propagator) due to the renormalization mechanism. More precisely in these field theories the calculated renormalized physical quantities are supposed to become independent of the *cut-off*. The latter must disappear in the transition from regularization to renormalization. Consequently the λ's remain in a finite fixed number and tend to definite (real) values as the *cut-off* cancels out. Within this scenario the poles representing physical masses (3 in our case) remain stable (even if shifted partly with respect to those computed approximately by us), because of to the assumed analytical properties of the renormalized propagator [12, 14].

VII. CONCLUSIONS

We have proposed a modification of the classical relativistic hamiltonian that allows the presence of several masses without changing its basic structure. This modification does not affect the infinite divisibility of the laws that are at the basis of the correspondence between stochastic processes and Lévy–quantum mechanics equations. However we discovered that the mentioned modification suggests a reformulation of the relativistic equations for wave functions and propagators in such a way that a suitable choice of the background noise produces a convergence in the perturbative contributions. To this purpose we remarked that a modification – with respect to the one given by equation (10) – of the logarithmic characteristic $\eta(\boldsymbol{u})$ by the insertion of the cut-off $f(x)$ allows to proceed to Regularization first, and then Renormalization of the two-point function of QCD. There are three parameters in our phenomenological $f(x)$ which is a third degree polynomial; the latter appears as the simplest choice that produces convergence in the integrals representing high order contributions to the fermion and boson propagators. Such parameters create three different poles in the zero-order propagators and allow the interpretation of a physical system with three different masses under precise constraints on $f(x)$. The masses might be related to the three families of the Standard Model. We like to point out that from the analyticity properties of the renormalized propagators, the mentioned poles tend to stabilize in the limit toward the complete solution (even if shifted with respect to the zero order ones) whereas the cut-off is expected to disappear because of the regular renormalizable QCD theory [12].

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