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Lévy processes and Schrödinger equation

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ABSTRACT

We analyze the extension of the well known relation between Brownian motion and the Schrödinger equation to the family of the Lévy processes. We consider a Lévy–Schrödinger equation where the usual kinetic energy operator – the Laplacian – is generalized by means of a selfadjoint, pseudodifferential operator whose symbol is the logarithmic characteristic of an infinitely divisible law. The Lévy–Khintchin formula shows then how to write down this operator in an integro-differential form. When the underlying Lévy process is stable we recover as a particular case the fractional Schrödinger equation. A few examples are finally given and we find that there are physically relevant models – such as a form of the relativistic Schrödinger equation – that are in the domain of the non stable Lévy–Schrödinger equations.

1. Introduction

That the Schrödinger equation can be linked to some underlying stochastic process has been well known for a long time. This idea has received along the years a number of different formulations: from the Feynman path integral [1], through the Bohm–Vigier model [2], to the Nelson stochastic mechanics [3,4]. In all these models the underlying stochastic process powering the random fluctuations is a Gaussian Brownian motion, and the focus of interest is the (non relativistic) Schrödinger equation of quantum mechanics. This particular choice is understandable because on one hand the Gaussian Brownian motion is the the most natural and widely explored example of Markov process available, and on the other hand its connection with the Schrödinger equation has always lent the hope of a finer understanding of quantum mysteries.

In the framework of stochastic mechanics, however, this standpoint can be considerably broadened since in fact this theory is a model for systems more general than quantum mechanics: a *dynamical theory of Brownian motion* that can be applied to several physical problems [5–7]. On the other hand in recent years we have witnessed a considerable growth of interest in non Gaussian stochastic processes, and in particular in the Lévy processes [8–12]. This is a field that was initially explored in the 30's and 40's of last century [13–15], but that achieved a full blossoming of research only in the last decades, also as a consequence of the tumultuous development of computing facilities. This interest is witnessed by the large field of the possible applications of these more general processes from statistical mechanics [7] to mathematical finance [10,16,17]. In the physical field, however, the research scope is presently rather confined to a particular kind of Lévy processes: the stable processes and the corresponding fractional calculus [18,19], while in the financial domain a vastly more general type of processes is at present in use. For instance also recently [20] the possibility of widening the perspective of the Schrödinger–Brownian pair has been considered, but that has been confined only to a fractional Schrödinger equation. The association of the more general Lévy infinitely divisible processes to the Schrödinger equation, instead, has been recognized

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as an important tool only in precious few papers [21,22] where the main interest of the authors was focused on the probabilistic interpretation of the relativistic quantum equations. Here, on the other hand, we suggest that the stochastic mechanics should be considered as a dynamical theory of the infinitely divisible processes showing time reversal invariance, and that the horizon of its applications should be widened even to cases different from quantum systems (see for example [5,6]).

This paper is devoted to a discussion of a generalization of the Schrödinger equation which takes into account the entire family of the Lévy processes: we will propose an equation where the infinitesimal generator of the Brownian semigroup (the Laplacian) is substituted by the more general generator of a Lévy semigroup. As it happens this will be a pseudodifferential operator (as, in particular, in the fractional case), and the Lévy–Khintchin formula will give us the opportunity to write it down in the form of an explicit integro-differential operator by putting in evidence its continuous (Gaussian) and its jumping (non Gaussian) parts. It is important to recall indeed that all the non Gaussian Lévy processes are characterized by the fact that their trajectories make jumps: a feature that can help to explain particular physical phenomena, as for instance the halo formation in intense charged particle beams in the accelerators [5,6], and the relativistic quantum mechanics [21,22]. The advantages of this formulation are many: first of all the widening of the increment laws from the stable to the infinitely divisible case will offer the possibility of having realistic, finite variances. As we will discuss later indeed, while all the non Gaussian, stable distributions have divergent variances — so that the range of the x decay rates of the stable probability density functions cannot exceed x^{-3} — this is not the case for the more general infinite divisible laws. On the other hand both the possible presence of a Gaussian component in the Lévy–Khintchin formula, and the wide spectrum of decay velocities of the increment probability densities will afford the possibility of having models with differences from the usual Brownian (and usual quantum mechanical, Schrödinger) case as small as we want. In this sense we could speak of small corrections to the quantum mechanical, Schrödinger equation. Last but not least, there are examples of non stable Lévy processes which are connected to a particular form of the quantum, relativistic Schrödinger equation: an important link that was missing in the original Nelson model. It seems in fact [21,22] that we can only recover some kind of relativistic quantum mechanics if we widen the field of the underlying stochastic processes at least to that of the selfdecomposable, jumping Lévy processes. To avoid formal complications we will confine our discussion to the case of processes in just one spatial dimension: generalizations will be straightforward.

2. A heuristic discussion

Let us start from the non relativistic, free Schrödinger equation associated with its propagator or Green function $G(x,t|y,s)$ (see for example Ref. [1])

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\,\partial_x^2\psi(x,t) \tag{1}$$

$$G(x,t|y,s) = \frac{1}{\sqrt{2\pi\,i(t-s)\hbar/m}}\,e^{-\frac{(x-y)^2}{2i(t-s)\hbar/m}} \tag{2}$$

$$\psi(x,t) = \int_{-\infty}^{+\infty} G(x,t|y,s)\,\psi(y,s)\mathrm{d}y \tag{3}$$

and compare it with the Fokker–Planck equation of a Wiener process (Brownian motion) with diffusion coefficient D, *pdf* (probability density function) $q(x,t)$ and transition *pdf* $p(x,t|y,s)$ (see for example Ref. [23])

$$\partial_t q(x,t) = D\,\partial_x^2 q(x,t) \tag{4}$$

$$p(x,t|y,s) = \frac{1}{\sqrt{4\pi(t-s)D}}\,e^{-\frac{(x-y)^2}{4(t-s)D}} \tag{5}$$

$$q(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,s)q(y,s)\mathrm{d}y. \tag{6}$$

It is apparent that there is a simple, formal procedure transforming the two structures one into the other:

$$D = \frac{\hbar}{2m},\quad t \longleftrightarrow it$$

It is well known that this is just the result of a time analytic continuation in the complex plane. There are of course important differences between G and p. For example while p and q are well behaved *pdf*'s, G is not a wave function, as can be seen also from a simple dimensional argument. This simple symmetry can then be deceptive, and a better understanding of its true meaning can be achieved either by means of the Feynman path integration with a free Lagrangian of the usual quadratic form, or through the Madelung decomposition [24] of (1) and its subsequent stochastic mechanical model [3,4]. Either way produces in the end the well known association between the Schrödinger equation and a very special form of background

Lévy noise: the (Gaussian) Brownian motion. Our aim here is to generalize to distributions other than Gaussian this simple shortcut from Wiener process to Schrödinger equation, and to analyze its most immediate consequences.

Let us see first of all what kind of role the gaussian distribution plays in our Wiener–Schrödinger scheme. The *pdf* and the *chf* (characteristic function) of a centered Gaussian law $\mathcal{N}(0, a^2)$

$$q(x) = \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}, \qquad \varphi(u) = e^{-a^2u^2/2}$$

satisfy the reciprocal relations

$$\varphi(u) = \int_{-\infty}^{+\infty} q(x)\, e^{iux}\, dx, \quad q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(u)\, e^{-iux}\, du. \tag{7}$$

Then the propagator (2) and the transition *pdf* (5) respectively have as *chf*'s (here $a^2 = 2D\tau$)

$$e^{-iD(t-s)u^2} = [\varphi(u)]^{i(t-s)/\tau}, \qquad e^{-D(t-s)u^2} = [\varphi(u)]^{(t-s)/\tau}$$

where now $\varphi(u) = e^{-D\tau u^2} = e^{-\tau\hbar u^2/2m}$ is the *chf* of a Gaussian law $\mathcal{N}(0, 2D\tau)$, and τ is a time constant introduced in order to have dimensionless exponents. From (7) we then have

$$G(x, t|y, s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\varphi(u)]^{i(t-s)/\tau}\; e^{-iu(x-y)}du \tag{8}$$

$$p(x, t|y, s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [\varphi(u)]^{(t-s)/\tau}\; e^{-iu(x-y)}du. \tag{9}$$

In both cases the starting point is the same *chf* φ of the centered, normal law $\mathcal{N}(0, 2D\tau)$. Then we consider the *chf* $[\varphi(u)]^{(t-s)/\tau}$ of the $t - s$ stationary increments of the Wiener process, we pass to the imaginary time variables ($t \leftrightarrow it$), and finally we get from (8) the Schrödinger propagator (2). The time scale τ incorporated in the initial normal law disappears in the subsequent steps when we generate the *chf* of the increments. This feature is common to all the stable laws and is the embodiment of the stable processes selfsimilarity.

The Eqs. (1) and (4) can now be easily deduced respectively from (8) and (9). For instance from (3) and (8) we have

$$\psi(x, t) = \int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} e^{-iDu^2(t-s)}e^{-iu(x-y)}du$$

and then we can write

$$\begin{aligned}
i\partial_t\psi(x, t) &= \int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} Du^2 e^{-iDu^2(t-s)}e^{-iu(x-y)}du \\
&= D\int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} (i\partial_x)^2 e^{-iDu^2(t-s)}e^{-iu(x-y)}du \\
&= -D\,\partial_x^2\int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} e^{-iDu^2(t-s)}e^{-iu(x-y)}du = -D\,\partial_x^2\psi(x, t)
\end{aligned}$$

which – but for a factor $\hbar$, and with $D = \hbar/2m$ – is the free, non relativistic Schrödinger equation (1).

We are now interested in reproducing these well known steps starting with the *chf* of a non Gaussian law. Take now an *infinitely divisible* – in general non Gaussian – law with *chf* $\varphi(u)$, and let $\eta(u) = \ln\varphi(u)$ be its *lch* (logarithmic characteristic). *Infinite divisibility* essentially is the property of a *chf* φ which guarantees that, for every real t, also $\varphi^{t/\tau}$ is a legitimate *chf*. About the infinitely divisible laws and their intimate relation with the Lévy processes there is a vast literature (see for example [8,15], and for a short introduction [25]). In the following we will restrict us to centered laws, and we will justify this choice in the subsequent sections. The law of the increment of the corresponding Lévy process then is $[\varphi(u)]^{(t-s)/\tau}$ and its transition *pdf* is (9) with our – possibly non Gaussian – infinitely divisible *chf*. Then, following the procedure previously outlined for the Wiener–Schrödinger equation, the wave function propagator is (8) with our new φ, and hence from (3) the time evolution is ruled by

$$\psi(x, t) = \int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} [\varphi(u)]^{i(t-s)/\tau}\, e^{-iu(x-y)}du.$$

The differential equation can then be deduced as in the Gaussian case and is

$$\begin{aligned}
i\partial_t\psi(x, t) &= \int_{-\infty}^{+\infty} dy\, \frac{\psi(y, s)}{2\pi}\int_{-\infty}^{+\infty} -\frac{\ln[\varphi(u)]}{\tau}\; [\varphi(u)]^{i(t-s)/\tau}\, e^{-iu(x-y)}du \\
&= -\frac{1}{\tau}\,\eta(\partial_x)\psi(x, t)
\end{aligned} \tag{10}$$

where now $\ln[\varphi(\partial_x)] = \eta(\partial_x)$ is a *pseudodifferential operator* with symbol $\eta(u)$ that is defined through the use of the Fourier transforms [10,11,26,27]. A pseudodifferential operator L on a suitable set of functions $h(x)$, is associated with a function $\ell(u)$ called the *symbol* of L (we indeed symbolically also write $L = \ell(\partial_x)$), and operates in the following way: if the Fourier transform of h is

$$\widehat{h}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x)\, e^{-iux} dx \tag{11}$$

then

$$(Lh)(x) = \ell(\partial_x)h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \ell(u)\, \widehat{h}(u)\, e^{iux} du. \tag{12}$$

Remark that the definition (11) of Fourier transform of a function $h(x)$ is traditionally slightly different from the definition (7) of the *chf* of a law. This definition entails that the norm of a square integrable function is preserved in the transformation: when dealing with Schrödinger equations this means that the Fourier transforms of normalized wave functions still are normalized wave functions for the conjugate observables. When the symbol is a polynomial of degree n

$$\ell(u) = \sum_{k=1}^{n} a_k (iu)^k$$

then L is a simple differential operator of order n

$$L = \sum_{k=1}^{n} a_k \partial_x^k$$

as can be easily seen from the properties of the Fourier transforms. However, even if $\ell(u)$ is not a polynomial, Eq. (12) defines an operator which is called pseudodifferential. We will now analyze the properties and the role of our pseudodifferential operator $\eta(\partial_x)$ to see if (10) can reasonably be considered as a generalized Schrödinger equation.

3. Semigroups and generators

Let $X(t)$ be a one dimensional Lévy process, namely a process with stationary and independent increments, and $X(0) = 0$ almost surely. The *chf* of its increments on a time interval Δt then is $[\varphi(u)]^{\Delta t/\tau}$ where $\varphi(u)$ is an infinitely divisible law, and τ a time scale parameter (see for example Refs. [8,11] for details about Lévy processes). It is well known that $\eta(u) = \ln \varphi(u)$ is the *lch* of an infinitely divisible law if and only if it satisfies the Lévy–Khintchin formula [11]

$$\eta(u) = i\gamma u - \frac{\beta^2}{2} u^2 + \int_{\mathbb{R}} \left[e^{iux} - 1 - iuxI_{[-1,1]}(x) \right] \nu(dx) \tag{13}$$

where $\gamma, \beta \in \mathbb{R}$, I_A is the indicator 0-1 function of the set A, and $\nu(\cdot)$ is the Lévy measure of our infinitely divisible law, namely a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(dx) < +\infty.$$

The integrals involving the Lévy measure ν should then in general be calculated on $\mathbb{R} - \{0\}$ to take into account its behavior near $y = 0$. The triplet (γ, β, ν) completely determines the Lévy process and is also called its *characteristic triplet*. There are a few equivalent formulations of this important result [8]. In particular the truncation function $I_{[-1,1]}(x)$ can be chosen in several different ways; this choice, however, will affect only the value of γ, while β and ν would be left unchanged.

To every Lévy process is associated a semigroup $(T_t)_{t\geq 0}$ acting on the space $\mathcal{D}$ of the measurable, bounded functions [11]: if $f \in \mathcal{D}$, we have $(T_t f)(x) = \mathbf{E}[f(X(t) + x)]$, where $\mathbf{E}$ is the expectation. The infinitesimal generator A of the semigroup (see Ref. [11] p. 131) is now defined on the domain of the functions $f \in \mathcal{D}$ such that the limit in norm

$$Af = \lim_{t\to 0^+} \frac{T_t f - f}{t}$$

exists. It can be proved (see Ref. [11] p. 139) that the generators of a Lévy process are pseudodifferential operators that can be extended to the Schwartz space $\mathcal{S}$ of the rapidly decreasing functions. In particular it can be proved that the symbol of A is nothing else than $\eta(u)$, namely $A = \eta(\partial_x)$, and that from the Lévy–Khintchin formula (13) we have

$$[\eta(\partial_x)f](x) = \gamma(\partial_x f)(x) + \frac{\beta^2}{2}(\partial_x^2 f)(x) + \int_{\mathbb{R}} \left[f(x+y) - f(x) - y(\partial_x f)(x) I_{[-1,1]}(y) \right] \nu(dy).$$

In other words our pseudodifferential operator $\eta(\partial_x)$ of Eq. (10) is the generator of the underlying Lévy process, and thanks to the Lévy–Khintchin formula it also has an explicit expression in terms of integro-differential operators.

The generator $A = \eta(\partial_x)$ can finally be extended to $L^2(\mathbb{R})$ which is a Hilbert space, so that we can also discuss its self-adjointness. In particular if $X(t)$ is a Lévy process, then its infinitesimal generator $A = \eta(\partial_y)$ will be self-adjoint in $L^2(\mathbb{R})$ if and only if $X(t)$ is centered and symmetric, namely if the symbol $\eta(u)$ is real with

$$\eta(u) = -\frac{\beta^2}{2}u^2 + \int_{\mathbb{R}} (\cos ux - 1)\,\nu(\mathrm{d}x) \tag{14}$$

where $\nu(\cdot)$ is a symmetric Lévy measure (see Ref. [11] p. 154). A Lévy measure is symmetric when $\nu(B) = \nu(-B)$ for every Borel measurable set, where $-B = \{x; -x \in B\}$. As a consequence the self-adjoint generators of the centered and symmetric Lévy processes enjoy the following simplified, integro-differential form

$$(Af)(x) = [\eta(\partial_x)f](x) = \frac{\beta^2}{2}\left(\partial_x^2 f\right)(x) + \int_{\mathbb{R}} [f(x+y) - f(x)]\,\nu(\mathrm{d}y). \tag{15}$$

It is also possible to show that $-\eta(\partial_x)$ is not only self-adjoint, but also positive on $L^2(\mathbb{R})$ in the sense that for every $f \in L^2(\mathbb{R})$ we have $-(f, \eta(\partial_x)f) \geq 0$ where $(\cdot,\cdot)$ is the usual scalar product on $L^2(\mathbb{R})$, and this is equivalent to say that the spectrum of $-\eta(\partial_x)$ lies entirely in $[0, +\infty)$.

We come back now to our proposed Lévy–Schrödinger equation. Let $X(t)$ be a centered, symmetric Lévy process with *pdf* and transition *pdf*: we know that there is a centered, infinitely divisible law with *chf* $\varphi(u) = \mathrm{e}^{\eta(u)}$ such that the *chf* of the stationary increments $\Delta X(t) = X(t + \Delta t) - X(t)$ is

$$[\varphi(u)]^{\Delta t/\tau} = \mathrm{e}^{\eta(u)\Delta t/\tau}$$

for a suitable time scale parameter τ. The transition *pdf* then is (9), so that by means of the substitution $t \leftrightarrow \mathrm{i}t$ we get the propagator G of Eq. (8). A wave function ψ then evolves following (10), and since our process is centered and symmetric the generator $\eta(\partial_x)$ is self-adjoint and has the integro-differential expression (15). That means that our proposed equation takes the form

$$\mathrm{i}\partial_t\psi(x,t) = -\frac{\eta(\partial_x)}{\tau}\psi(x,t) = -\frac{\beta^2}{2\tau}\,\partial_x^2\psi(x) - \frac{1}{\tau}\int_{\mathbb{R}} [\psi(x+y) - \psi(x)]\,\nu(\mathrm{d}y). \tag{16}$$

When $X(t)$ is a Gaussian Wiener process we know that $\eta(u) = -\beta^2u^2/2$ and $A = \eta(\partial_x) = \frac{\beta^2}{2}\partial_x^2$. Then the process evolution equation is reduced to the Fokker–Planck equation (4) with $D = \beta^2/2\tau$, and in this case (16) coincides with the usual non relativistic, free Schrödinger equation. In fact this amounts to take the – self-adjoint and positive in $L^2(\mathbb{R})$ – (pseudo) differential operator $-\hbar\eta(\partial_x)/\tau = -\hbar D\partial_x^2$ as the kinetic energy operator. We propose here to extend this association also to the case of non Gaussian Lévy processes. In fact the same kind of association has been proposed for the fractional Schrödinger equations [20], albeit in a far more restricted perspective: that of the stable laws, which are a particular class of infinitely divisible laws (see Section 4.3).

Notice that we will adopt here only the substitution $t \leftrightarrow \mathrm{i}t$, but we will not impose $D = \hbar/2m$ because our model is not necessarily supposed to be some kind of generalized quantum mechanics, but will rather describe a *dynamical theory of Lévy processes* in the spirit of Nelson stochastic mechanics. If we introduce a constant α with the dimensions of an action our proposed free Lévy–Schrödinger equation (16) becomes

$$\begin{aligned}\mathrm{i}\alpha\partial_t\psi(x,t) &= H_0\psi(x,t) = -\frac{\alpha}{\tau}\,\eta(\partial_x)\psi(x,t)\\ &= -\alpha\frac{\beta^2}{2\tau}\,\partial_x^2\psi(x,t) - \frac{\alpha}{\tau}\int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)]\,\nu(\mathrm{d}y)\end{aligned} \tag{17}$$

where now the free hamiltonian operator H_0 has the dimensions of an energy. This integro-differential hamiltonian H_0 is self-adjoint and positive on $L^2(\mathbb{R})$ so that it is a good kinetic energy operator. Everything that can be deduced about the usual Schrödinger equation from the positivity and self-adjointness of H_0 can also be of course extended to our Lévy–Schrödinger equation (17). In particular the conservation of the probability in the sense that, if $|\psi|^2$ plays the role of the position *pdf*, then the norm $\|\psi\|^2$ will be constant. Because of the self-adjointness of $\eta(\partial_x)$ will have indeed from (17)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\psi(t),\psi(t)) = (\partial_t\psi,\psi) + (\psi,\partial_t\psi) = \frac{(\eta(\partial_x)\psi,\psi) - (\psi,\eta(\partial_x)\psi)}{\mathrm{i}\tau} = 0.$$

We could finally add a potential $V(x)$ to (17) and get also a complete Lévy–Schrödinger equation

$$\mathrm{i}\alpha\partial_t\psi(x,t) = H\psi(x,t) = -\frac{\alpha}{\tau}\,\eta(\partial_x)\psi(x,t) + V(x)\psi(x,t) \tag{18}$$

where the hamiltonian is now $H = H_0 + V$.

4. Discussion and examples

The free Lévy–Schrödinger hamiltonian of equation (17) contains two parts: the usual kinetic energy $-\frac{\alpha\beta^2}{2\tau}\,\partial_x^2$ related to the Gaussian part of the process; and the jump part which is given by means of an integral with a symmetric Lévy measure ν. Of course, depending on the nature of the underlying process, the Eq. (17) can contain these components in different mixtures. If the underlying process is purely Gaussian then the Lévy measure ν vanishes, almost every trajectory of the underlying process will be continue, and (17) is reduced to the usual Schrödinger equation. To the other end of the gamut, when the underlying noise has no Gaussian component its trajectories make jumps, $\beta = 0$, and we get a pure jump Lévy–Schrödinger equation. In general both terms are present and, if for instance we introduce $\omega = 1/\tau$ and choose $\alpha = \hbar$ and $\beta^2 = \alpha\tau/m$, then (17) takes the form

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\,\partial_x^2\psi(x,t) - \hbar\omega\int_{\mathbb{R}}[\psi(x+y,t)-\psi(x,t)]\,\nu(dy).$$

Here, for example, the integral jump term can be considered as a correction to the usual Schrödinger equation and its weight, the energy $\hbar\omega$, is at present a free parameter. In particular for $\beta = 0$ we get pure jump Schrödinger equations of the form

$$i\partial_t\psi(x,t) = -\omega\int_{\mathbb{R}}[\psi(x+y,t)-\psi(x,t)]\,\nu(dy). \tag{19}$$

Of course these remarks emphasize the fact that the explicit form of the Lévy–Schrödinger equation will depend on the choice of the particular Lévy measure ν.

4.1. Comparison with other approaches

We have already recalled in the Section 1 that in the last few years several other lines of research have associated the Lévy processes with the Schrödinger equation. For instance in a series of papers [20] Laskin has developed a fractional Schrödinger equation and the corresponding fractional quantum mechanics. As we will see in the subsequent Section 4.3 the fractional calculus [19] needed to this approach is related to the use of background noises generated by stable Lévy laws, an important subclass of the more general infinitely divisible laws.

The appeal of the stable distributions is justified by the properties of scaling and self–similarity displayed by the corresponding processes, but it must also be remarked that these distributions show a few features that partly impair their usefulness as empirical models. First of all the non gaussian stable laws always have infinite variance. Then the range of the x decay rates of the probability density functions cannot exceed x^{-3}, and this too introduces a particular rigidity in these models. This makes them rather suspect as a realistic tool and prompts the introduction either of fractional measures of the distribution dispersion, or of *truncated* stable distributions which, however, are no longer stable. On the other hand the divergence of the second moment means that these processes represent signals with infinite average instantaneous power, and – from a mathematical standpoint – that they are not square integrable processes so that they cannot take advantage of all the geometrical structures present in an L^2 Hilbert space.

The more general Lévy processes instead are generated by infinitely divisible laws and do not necessarily show these disturbing features, but they can be more difficult to analyze and to simulate (as for example in the case of the Student process [28–30]). Besides the fact that they do not have natural scaling properties, the probability density function of their increments could be explicitly known only at one time instant. In fact, while their time evolution can always be explicitly given in terms of characteristic functions, their marginal densities may not be calculable. This is a feature, however, that they share with most stable processes, since the probability density functions of the non gaussian stable laws are explicitly known in terms of elementary functions only in few cases. On the other hand some new applications in the physical domain for Lévy, infinitely divisible but not stable processes begin to emerge: in particular the statistical characteristics of some recent model of the collective motion in the charged particle accelerator beams seem to point exactly in the direction of some kind of Student infinitely divisible process [6,31]. It should be stressed again, moreover, that there is another physically relevant case that can be associated with a Lévy non stable noise: that of the relativistic Schrödinger equation. This apparently shows that there are significant models to be found outside the domain of the more popular stable distributions.

The relevance of this last remark, on the other hand, has been well understood by De Angelis [21] and by Garbaczewski and his coworkers [22] who in several papers have analyzed the possibility of using a wider spectrum of infinitely divisible distributions and of the corresponding Schrödinger equations. In particular the focus of their interest has been the relativistic Schrödinger equation that – as will be analyzed in the subsequent sections – can be associated to a subclass of selfdecomposable laws. The interest of this choice is apparent since it is connected to the well known problem of the single–particle, probabilistic interpretation of the Klein–Gordon equation. On the other hand the authors in Ref. [22] again and again point out that their approach is not coincident with that of the Nelson stochastic mechanics, but rather is inscribed in what is known as the *Schrödinger problem*: that of deducing the probabilistic interpolation (a stochastic process also known as *Schrödinger bridge*, see for example Ref. [32] and references quoted therein) consistent with a given pair of boundary measure data at a fixed initial and terminal time instants $t_1 < t_2$. In this framework Garbaczewski analyzes several questions connected with the markovianity of the interpolating process and, to do that, he takes a special care in elaborating explicit examples of a Cauchy–Schrödinger dynamics. As it is well known the Cauchy laws are the most tractable case of non Gaussian, stable laws, and hence the generators of the associated Lévy processes fall in the class of the pseudodifferential, fractional operators mentioned earlier. Remember also that the Cauchy law is the prototype of the non Gaussian, stable laws with infinite variance. It is important then to extend our stochastic analysis also to the classes of non stable, infinitely divisible laws, and in the subsequent sections we will give an overview of the possible, calculable examples of these laws beyond the Cauchy and the Relativistic case. In particular we will focus our attention on a few explicit examples of laws, namely the Variance Gamma, the Student and the compound Poisson laws which are not considered in the previously quoted papers. On the other hand the mentioned possible applications of the Lévy–Schrödinger equation in domains different from the quantum mechanics, will qualify this model as a general dynamical theory of the Lévy processes, in the same spirit of the Nelson stochastic mechanics.

In the Laskin papers [20], finally, a generalized form of Feynman path integrals has been used to discuss the meaning of the fractional propagators: in particular the amplitudes associated with the Feynman paths have been generalized by substituting the non Gaussian, stable *pdf*'s to the usual Gaussian laws. In the momentum representation, for example, the fractional kernel has just a term $|p|^\lambda$ $(0 < \lambda \leq 2)$ instead of the term p^2 of the Feynman kernel: namely the Gaussian *lch* u^2 has been replaced by the stable $|u|^\lambda$ (see Section 4.3). This line of reasoning can entirely be transferred here, because our generalized, infinitely divisible propagators (8) can also be expressed in term of Feynman integrals by generalizing the path amplitudes to infinitely divisible *pdf*'s. In particular the stable *lch* $|u|^\lambda$ will just be replaced by a generalized infinitely divisible $\eta(u)$. The very nature of the underlying Lévy processes would entail the coherence of this approach. Here however we will not elaborate this point and we will take for granted this association between generators and kinetic energy operators in order to establish the most general form of the free Lévy–Schrödinger equation and then discuss its first consequences.

4.2. Stationary free solutions

Let us consider first the stationary solutions of (17): taking

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}Et/\alpha}\phi(x), \qquad \mathrm{i}\alpha\partial_t\psi(x,t) = E\psi(x,t)$$

we have that the spatial part $\phi(x)$ will be solution of

$$H_0\phi(x) = -\frac{\alpha\beta^2}{2\tau}\,\phi''(x) - \frac{\alpha}{\tau}\int_{\mathbb{R}}[\phi(x+y)-\phi(x)]\,\nu(\mathrm{d}y) = E\phi(x). \tag{20}$$

For the plane wave solutions

$$\phi(x) = \mathrm{e}^{\pm\mathrm{i}ux}$$

and because of the symmetry of the Lévy measure ν, Eq. (20) becomes

$$\begin{aligned} E\phi(x) &= \left[-\frac{\alpha\beta^2}{2\tau}\,u^2 - \frac{\alpha}{\tau}\int_{\mathbb{R}}(\mathrm{e}^{\pm iuy}-1)\,\nu(\mathrm{d}y)\right]\phi(x) \\ &= \frac{\alpha}{\tau}\left[\frac{\beta^2u^2}{2} - \int_{\mathbb{R}}(\cos uy - 1)\,\nu(\mathrm{d}y)\right]\phi(x) = -\frac{\alpha}{\tau}\eta(u)\phi(x) \end{aligned}$$

and hence it is satisfied when between E and u the following relation holds

$$E = -\frac{\alpha}{\tau}\eta(u).$$

Here u is a wave number, while we are used to look for a relation between energy E and momentum p. If then we posit $p = \alpha u$ the energy–momentum relation for our free Lévy–Schrödinger equation is

$$E = -\frac{\alpha}{\tau}\,\eta\left(\frac{p}{\alpha}\right). \tag{21}$$

4.3. Some classes of infinitely divisible laws

We will explore in the subsequent sections a few examples of Lévy–Schrödinger equations associated with Lévy processes. For a short summary of the concepts used here see for example Ref. [25]. To begin with we will consider the *chf*'s, Lévy measures and infinitesimal generators of centered, symmetric, infinitely divisible laws so that (14) and (15) hold. The form of the Lévy measure ν is here instrumental to explicitly show how the pseudo-differential generator $\eta(\partial_x)$ works. It would then be useful to list several classes of infinitely divisible laws in a growing order of generality:

1. *Stable laws*: here we have [8,10]

$$\eta(u) = -\frac{(a|u|)^\lambda}{\lambda}; \quad 0 < \lambda \leq 2, \tag{22}$$

with the important particular cases

$$\eta(u) = \begin{cases} -a^2u^2/2 & \text{Gauss law } (\lambda = 2); \\ -a|u| & \text{Cauchy law } (\lambda = 1). \end{cases}$$

Stable laws are selfdecomposable and hence their Lévy measures are absolutely continuous [8,10] so that $\nu(\mathrm{d}x) = W(x)\,\mathrm{d}x$ with Lévy density

$$W(x) = \frac{B}{|x|^{\lambda+1}}, \qquad B = \begin{cases} 0 & \lambda = 2; \\ a/\pi & \lambda = 1; \\ -\dfrac{a^\lambda}{2\lambda\,\cos\frac{\lambda\pi}{2}\,\Gamma(-\lambda)} & \lambda \neq 1, 2. \end{cases} \tag{23}$$

The infinitesimal generator, which in the Gauss case ($\lambda = 2$) simply is

$$\frac{a^2}{2}(\partial_x^2 f)(x),$$

for $0 < \lambda < 2$ becomes the pseudo-differential operator

$$[\eta(\partial_x)f](x) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{(a|u|)^\lambda}{\lambda}\,\mathrm{e}^{\mathrm{i}ux}\widehat{f}(u)\,\mathrm{d}u = B\int_{-\infty}^{+\infty}\frac{f(x+y)-f(x)}{|y|^{\lambda+1}}\,\mathrm{d}y$$

which can also be symbolically expressed in terms of the *fractional derivatives* [19]

$$\begin{aligned}(\partial_x^\lambda f)(x) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|u|^\lambda \mathrm{e}^{\mathrm{i}ux}\widehat{f}(u)\,\mathrm{d}u \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(-\partial_x^2)^{\lambda/2}\mathrm{e}^{\mathrm{i}ux}\widehat{f}(u)\,\mathrm{d}u = \left[-(-\partial_x^2)^{\lambda/2}f\right](x).\end{aligned} \tag{24}$$

The Cauchy case is discussed also in Ref. [22].

2. *Selfdecomposable (non stable) laws*: they are an important sub-family of infinitely divisible laws. Two examples are (for details and other examples see Refs. [25,29])

$$\eta(u) = \begin{cases} -\lambda\,\ln(1+a^2u^2), & \text{Variance-Gamma } (\lambda > 0); \\ 1-\sqrt{1+a^2u^2}, & \text{Relativistic quantum mechanics} \end{cases}$$

which have no Gaussian part ($\beta = 0$ in the Lévy–Khintchin formula) and produce pure jump processes. Their Lévy measures have densities [25,29]

$$W(x) = \begin{cases} \lambda|x|^{-1}\mathrm{e}^{-|x|/a} & \text{VG}; \\ (\pi|x|)^{-1}K_1(|x|/a) & \text{Relativistic q.m.} \end{cases}$$

where $K_\lambda(z)$ is a modified Bessel function. Remark that, while for the VG we can explicitly write the *pdf*

$$q(x) = \frac{2}{a\,2^\lambda\Gamma(\lambda)\sqrt{2\pi}}\left(\frac{|x|}{a}\right)^{\lambda-\frac{1}{2}}K_{\lambda-\frac{1}{2}}\left(\frac{|x|}{a}\right)$$

we have no elementary expressions for the Relativistic q.m. *pdf*

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{1-\sqrt{1+a^2u^2}}\mathrm{e}^{\mathrm{i}ux}\mathrm{d}u$$

which can then be calculated only numerically. There is finally at least another important family of selfdecomposable laws that has attracted considerable attention in recent years [28–30] and deserves to be mentioned here: the Student laws. As is known their *lch*'s are not elementary ($\lambda > 0$)

$$\eta(u) = \ln\left[\frac{2(a|u|)^{\lambda/2}K_{\lambda/2}(a|u|)}{2^{\lambda/2}\Gamma(\lambda/2)}\right]$$

where $\Gamma(z)$ is the Euler Gamma function, but their *pdf* are comparatively simpler than the previous ones

$$q(x) = \frac{1}{a\,B(1/2,\ \lambda/2)}\left(1+\frac{x^2}{a^2}\right)^{-\lambda/2}$$

where $B(x, y)$ is the Euler Beta function. What makes the corresponding Lévy process (the Student process) a harder nut to crack than, for example, the VG process is the fact that, even if both the Student and the VG distributions are not stable, the family of the VG laws is closed under convolution, while the Student family is not. This means that in a VG process the marginals always are VG distributions, while in the Student case the marginals are explicitly known only at one time instant. It is worth noting, however, that to write down the corresponding Lévy–Schrödinger equation we do not use a detailed description of the underlying process: we just need to know the *lch* of the generating law $\eta(u)$ and its Lévy density $W(x)$. The Student process has been recently analyzed [29] in the case $\lambda = 3$ where the *lch* and the *pdf* have the elementary expressions

$$\eta(u) = -a|u| + \ln(1 + a|u|), \qquad q(x) = \frac{2}{a\pi}\left(1 + \frac{x^2}{a^2}\right)^{-2} \tag{25}$$

and the density of the Lévy measure has been explicitly calculated:

$$W(x) = \frac{a}{\pi x^2}\left[1 - \frac{|x|}{a}\left(\sin\frac{|x|}{a}\,\mathrm{ci}\frac{|x|}{a} - \cos\frac{|x|}{a}\,\mathrm{si}\frac{|x|}{a}\right)\right] \tag{26}$$

where ci(x) and si(x) are the cosine and sine integral functions

$$\mathrm{ci}(x) = -\int_x^{+\infty}\frac{\cos t}{t}\mathrm{d}t \qquad \mathrm{si}(x) = -\int_x^{+\infty}\frac{\sin t}{t}\mathrm{d}t.$$

3. *Infinitely divisible (non selfdecomposable) laws*: The classical example of an infinitely divisible, non selfdecomposable law is the Poisson law of intensity λ, but the corresponding Lévy process would not be symmetric. If however we take the *chf* $\chi(u)$ of a centered, symmetric law, then the corresponding *compound* Poisson process will be centered and symmetric with *lch*

$$\eta(u) = \lambda\,(\chi(u) - 1)\,.$$

In the analysis of the corresponding Lévy measure we must remember that now we can no longer take for granted that ν is absolutely continuous. For example if the jump size can take only two values $\pm a$ $(a > 0)$ with equal probabilities 1/2, then the *chf* $\chi(u) = \cos au$ has no *pdf*, $\eta(u) = \lambda(\cos au - 1)$, and $\nu(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ where the cumulative distribution

$$F(x) = \frac{\Theta(x-a) + \Theta(x+a)}{2} \tag{27}$$

is a symmetric, two-steps function, and $\Theta(x)$ is the 0–1 Heaviside function. If on the other hand χ is a law with a *pdf* $g(x)$, it is possible to show that also the Lévy measure ν is absolutely continuous with a density

$$W(x) = \lambda g(x).$$

This completely specify the associated Lévy process on the basis of the Poisson intensity λ, and of the law of the jump sizes.

4.4. Examples of Lévy–Schrödinger equations

We can now analyze a few examples of Lévy–Schrödinger equations based on the laws listed above, by putting in evidence their integro-differential form and the energy momentum relations associated to every particular choice.

1. *Non relativistic, quantum, free particle*: this is the well known case of the Gaussian Wiener process with $\eta(u) = -\beta^2u^2/2$ giving rise to the usual Schrödinger equation (1) for a suitable identification of the parameters. In this case the energy–momentum relation (21) is

$$E = -\frac{\alpha}{\tau}\left(-\frac{\beta^2}{2}\frac{p^2}{\alpha^2}\right) = \frac{\beta^2}{2\alpha\tau}p^2$$

and with $\alpha = \hbar$ and $\beta^2 = \alpha\tau/m$ we get as usual

$$E = \frac{p^2}{2m}, \quad p = \hbar u.$$

2. *Relativistic, quantum, free particle*: it is important to remark at this point that there is a pure jump Lévy process which is connected to the relativistic Schrödinger equation, in the same way as the Wiener process is connected to the non relativistic Schrödinger equation. Take the non stable, selfdecomposable law $\eta(u) = 1 - \sqrt{1 + a^2u^2}$, and use the following identifications

$$\frac{\alpha}{\tau} = mc^2, \qquad a = \frac{\hbar}{mc}, \qquad p = \hbar u$$

to find from (21)

$$E = -mc^2\eta(u) = \sqrt{m^2c^4 + p^2c^2} - mc^2 \tag{28}$$

which is the relativistic total energy less the rest energy mc^2: namely the kinetic energy. The corresponding Lévy–Schrödinger equation is now

$$\mathrm{i}\hbar\partial_t\psi(x,t) = \left[\sqrt{m^2c^4 - c^2\hbar^2\,\partial_x^2} - mc^2\right]\psi(x,t)$$

and it is discussed also in Ref. [22]. Since the constant $-mc^2$ can be reabsorbed by means of a phase factor $e^{imc^2t/\hbar}$, the wave equation finally is

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - c^2\hbar^2\partial_x^2}\,\psi(x,t) \tag{29}$$

which is the simplest form of a relativistic, free Schrödinger equation [33]. It is interesting to note that the Lévy process behind the relativistic equation (29) is a pure jump process with an absolutely continuous Lévy measure with *pdf*

$$W(x) = \frac{1}{\pi|x|}K_1\left(\frac{|x|}{a}\right) = \frac{1}{\pi|x|}K_1\left(\frac{mc|x|}{\hbar}\right)$$

so that Eq. (29) can also be written as

$$i\hbar\partial_t\psi(x,t) = -mc^2\int_{\mathbb{R}}\frac{\psi(x+y,t)-\psi(x,t)}{\pi|y|}K_1\left(\frac{mc|y|}{\hbar}\right)dy.$$

From the form of the relativistic energy (28) also the usual relativistic corrections to the classical energy–momentum relation for small values of p/c follow:

$$E = mc^2\left(\sqrt{1+\frac{p^2}{m^2c^2}}-1\right) = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + o(p^5).$$

Finally if we consider $E = H(p)$ as the Hamiltonian function of a relativistic free particle from the Hamilton equations we get

$$\dot{q} = \partial_p H = \frac{p}{m}\frac{1}{\sqrt{1+p^2/m^2c^2}}$$

and here too we can see the relativistic correction to the classical kinematic relation $p = m\dot{q}$.

3. *Variance-Gamma laws*: for the particular case $\lambda = \frac{1}{2}$ we have

$$\eta(u) = -\frac{1}{2}\ln(1+a^2u^2), \qquad q(x) = \frac{1}{a\pi}K_0\left(\frac{|x|}{a}\right), \qquad W(x) = \frac{e^{-|x|/a}}{2|x|}.$$

The Variance-Gamma processes are pure jump processes with no Gaussian part in the Lévy–Khintchin formula (13) ($\beta = 0$), so that the Lévy–Schrödinger equation becomes

$$i\alpha\partial_t\psi(x,t) = -\frac{\alpha}{\tau}\int_{\mathbb{R}}\frac{\psi(x+y,t)-\psi(x,t)}{2|y|}e^{-|y|/a}dy.$$

By choosing

$$\alpha = \frac{ma^2}{\tau}, \qquad p = \frac{ma^2}{\tau}u \tag{30}$$

we have the following energy–momentum relation (for $p\tau/ma \to 0$)

$$E = \frac{ma^2}{2\tau^2}\ln\left(1+\frac{\tau^2p^2}{m^2a^2}\right) = \frac{p^2}{2m} - \frac{\tau^2}{2ma^2}\frac{p^4}{m^2} + o(p^5)$$

while with the identification $E = H(p)$ we can also recover the kinematic relations between p and $\dot{q}$:

$$\dot{q} = \frac{p/m}{1+\frac{\tau^2p^2}{a^2m^2}} = \frac{p}{m} - \frac{\tau^2p^3}{a^2m^3} + o(p^4).$$

It is apparent then that again these equations are corrections to the classical relations.

4. *Student laws*: also the Student processes are pure jump processes with no Gaussian part in the Lévy–Khintchin formula, and for $\lambda = 3$ the *pdf*, *lch* and Lévy density are listed in (25) and (26), so that the Lévy–Schrödinger equation is

$$i\alpha\partial_t\psi(x,t) = -\frac{\alpha}{\tau}\int_{\mathbb{R}}\frac{\psi(x+y,t)-\psi(x,t)}{\pi y^2}a\left[1-\frac{|y|}{a}\left(\sin\frac{|y|}{a}\,\mathrm{ci}\frac{|y|}{a} - \cos\frac{|y|}{a}\,\mathrm{si}\frac{|y|}{a}\right)\right]dy.$$

If then we take (30) the energy–momentum relations become (with $p\tau/ma \to 0$)

$$E = \frac{a|p|}{\tau} - \frac{ma^2}{\tau^2}\ln\left(1+\frac{|p|\tau}{ma}\right) = \frac{p^2}{2m} - \frac{\tau|p|^3}{3m^2a} + o(|p|^3)$$

and the kinematical ones

$$\dot{q} = \frac{p/m}{1+\frac{\tau|p|}{am}} = \frac{p}{m} - \frac{|p|}{p}\frac{\tau p^2}{am^2} + o(p^2).$$

As in the case of all the other selfdecomposable laws, here too we find that these mechanical relations are just corrections to the classical equations when p is small with respect to ma/τ.

5. *Stable laws*: for the non Gaussian stable laws we have the *lch* (22) and the Lévy measure *pdf* (23) where $0 < \lambda < 2$. With a couple of exception (Cauchy and Lévy laws), however, there are no elementary formulas for their *pdf*'s $q(x)$. Their Lévy–Schrödinger equation is nothing else than the so called *fractional Schrödinger equation* [20] expressed also in terms of the fractional derivatives (24)

$$\mathrm{i}\alpha\partial_t\psi(x,t) = \frac{\alpha}{\tau}\,\partial_x^\lambda\psi(x,t) = \frac{\alpha}{\tau}\,\frac{a^\lambda}{2\lambda\cos\frac{\lambda\pi}{2}\Gamma(-\lambda)}\int_{\mathbb{R}}\frac{\psi(x+y,t)-\psi(x,t)}{|y|^{\lambda+1}}\,\mathrm{d}y$$

and with the identifications (30) its energy–momentum relations become

$$E = \frac{\alpha}{\tau}\,\frac{(a|u|)^\lambda}{\lambda} = \frac{2^{\lambda/2}}{\lambda}\left(\frac{ma^2}{\tau^2}\right)^{1-\lambda/2}\left(\frac{p^2}{2m}\right)^{\lambda/2}. \tag{31}$$

This however looks not as a correction of the classical formula as in the other cases considered, but rather as a completely different formula. The same can be said of the kinematic relations between $\dot{q}$ and p which are now

$$\dot{q} = \frac{p}{m}\left(\frac{p^2\tau^2}{m^2a^2}\right)^{\lambda/2-1}. \tag{32}$$

The relations (31) and (32) are still another reason to deem not advisable to restrict an inquiry on Lévy processes and Schrödinger equation only to the family of stable processes.

6. *Compound Poisson process*: for the symmetric, compound Poisson process with the Lévy measure (27) the pure jump Lévy–Schrödinger equation (19) greatly simplifies as

$$\mathrm{i}\partial_t\psi(x,t) = -\frac{\lambda\omega}{2}\left[\psi(x+a,t)-2\psi(x,t)+\psi(x-a,t)\right].$$

We can now show that the usual Schrödinger equation (1) can always be recovered as a limit case of this Poisson–Schrödinger equation. If ψ is twice differentiable in x, we know that for $a \to 0^+$ we have

$$\psi(x\pm a,t) = \psi(x,t) \pm a\partial_x\psi(x,t) + \frac{a^2}{2}\partial_x^2\psi(x,t) + o(a^2)$$

and hence

$$\mathrm{i}\partial_t\psi(x,t) = -\frac{\lambda\omega a^2}{2}\,\partial_x^2\psi(x,t) + \lambda\,o(a^2).$$

Now, if, as $a \to 0^+$, also $\lambda \to +\infty$ in such a way that $\lambda a^2 \to b^2$, then we have

$$\mathrm{i}\partial_t\psi(x,t) = -\frac{\omega b^2}{2}\,\partial_x^2\psi(x,t)$$

where $\omega b^2/2$ has the dimensions of a diffusion coefficient: namely, in the limit, we get a Wiener–Schrödinger equation of the type (1). This procedure, for instance, could also be used to introduce small corrections in the coefficient $\hbar^2/2m$ of a Schrödinger equation. Remark that this ability of the compound Poisson process to approximate Gaussian processes is only a consequence of a much more general behavior: in fact every Lévy process (not only the Gaussian one) can be pathwise approximated by a suitable sequence of compound Poisson processes (for details see Ref. [8] p. 342).

4.5. Perfectly rigid walls

An example of solution of the complete Lévy–Schrödinger equation (18) can easily be obtained in the case of a system confined between two perfectly rigid walls symmetrically located at $x = \pm L/2$. The discussion is similar to that of Section 4.2, but for the boundary conditions which now require that the solutions vanish at $x = \pm L/2$. As a consequence the solutions are the usual trigonometric functions with discrete eigenvalues

$$E_n = -\frac{\alpha}{\tau}\,\eta(u_n), \quad u_n = \frac{n\pi}{L},\; n = 1, 2, \ldots .$$

The form of the eigenvalue sequence will depend on the *lch* η. If the underlying process is a Wiener process we have $\eta(u) = -\beta^2u^2/2$, and hence

$$E_n = \frac{\alpha\beta^2\pi^2}{2\tau L^2}\,n^2$$

which, with the identifications $\alpha = \hbar$ and $\beta^2 = \alpha\tau/m$, coincides with the usual quantum mechanical result. On the other hand for an underlying Variance-Gamma noise we find

$$E_n = \frac{\lambda\alpha}{\tau} \ln\left(1 + \frac{a^2\pi^2}{L^2} n^2\right).$$

Finally for a symmetric, compound Poisson noise with intensity λ and equiprobable jump sizes $\pm a$ we get

$$E_n = \frac{\lambda\alpha}{\tau} \left(1 - \cos\frac{a\pi n}{L}\right).$$

Apparently this no longer is a monotone increasing sequence: E_n goes up and down between 0 and $2\alpha\lambda/\tau$. If a/L is rational the sequence is periodic; on the other hand when a/L is irrational there are no coincident eigenvalues in the sequence, so that the E_n will fill the bandwidth between 0 and $2\alpha\lambda/\tau$.

5. Conclusions

We have discussed the possibility of generalizing the relation between Brownian motion and the Schrödinger equation by associating the kinetic energy of a physical system with the generator of a symmetric Lévy process, namely to a pseudodifferential operator whose symbol is the *lch* of an infinitely divisible law. This amounts to suppose, then, that this new Lévy–Schrödinger equation is based on an underlying Lévy process that can have both Gaussian (continuous) and non Gaussian (jumping) components. In recent years other extensions of the Schrödinger equation have been put forward in the same spirit of our Lévy–Schrödinger equation. In particular we refer to several papers about fractional Schrödinger equations [20] that explored the use of fractional calculus in a kind of generalized quantum mechanics. As it is clear from the previous sections, however, this is the particular case when our underlying process is stable. The extension to the infinitely divisible, non stable processes, on the other hand, is physically meaningful because there are significant cases that are in the domain of our Lévy–Schrödinger picture, without being in that of the fractional Schrödinger equation. In particular, as shown also in Ref. [22], the simplest form of a relativistic, free Schrödinger equation can be associated with a particular type of selfdecomposable, non stable process acting as background noise. Moreover in many instances of the Lévy–Schrödinger equation the new energy–momentum relations can be seen as corrections to the classical relations for small values of certain parameters. It must be remembered moreover that our model is not tied to the use of processes with infinite variance: the variances can be chosen to be finite even in a purely non Gaussian model – as in the case of the relativistic, free Schrödinger equation – and can then be used as a legitimate measure of the dispersion. Finally let us recall that a typical non stable, Student Lévy noise seems to be suitable for applications in the models of halo formation in intense beam of charged particles in accelerators [6,29,31].

It is important now to explicitly give in full detail a discussion of the processes associated with the wave functions of the Lévy–Schrödinger equation, namely of a generalized stochastic mechanics. In our opinion this will be possible because the techniques of the stochastic calculus applied to Lévy processes are today in full development [8–12], and at our knowledge there is no apparent, fundamental impediment along this road. At present we have confined ourselves to give heuristic arguments based on both the identification of the self–adjoint process generators as the kinetic energy operators, and the usual analytic continuation of the time variable t to its imaginary counterpart it. In an equivalent vein it would be possible to follow the Feynman integral road in analyzing the propagators (8) in term of integrations over paths with amplitudes deduced now from some kind of infinitely divisible distribution. It will be important now try to generalize the Nelson stochastic mechanics by adding suitable dynamics to our Lévy processes, and by studying the Lévy–Nelson processes associated with the wave functions: this will be the subject of a future paper.

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