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Mixtures in nonstable Lévy processes

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Abstract
We analyse the Lévy processes produced by means of two interconnected classes of nonstable, infinitely divisible distribution: the variance gamma and the Student laws. While the variance gamma family is closed under convolution, the Student one is not: this makes its time evolution more complicated. We prove that—at least for one particular type of Student processes suggested by recent empirical results, and for integral times—the distribution of the process is a mixture of other types of Student distributions, randomized by means of a new probability distribution. The mixture is such that along the time the asymptotic behaviour of the probability density functions always coincide with that of the generating Student law. We put forward the conjecture that this can be a general feature of the Student processes. We finally analyse the Ornstein-Uhlenbeck process driven by our Lévy noises and show a few simulations of it.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recently the Lévy processes have enjoyed considerable popularity in several different fields of research from statistical physics to mathematical finance (Paul and Baschnagel (1999), Mantegna and Stanley (2001), Barndorff-Nielsen *et al* (2001) and Cont and Tankov (2004) are just a few examples of books reviewing the large body of literature on this subject). In the former field, however, the interest has been generally confined to α-stable processes which are an important particular sub-class of Lévy processes (Bouchaud and Georges 1990, Metzler and Klafter 2000, Paul and Baschnagel 1999, Woyczyński 2001), while studies about nonstable, infinitely divisible Lévy processes abound mainly in the latter field (see for example Cont and Tankov 2004 and references quoted therein). The appeal of the α-stable distributions is justified by the properties of scaling and self-similarity displayed by the corresponding processes, but it must also be remarked that these distributions show a few features that partly impair their usefulness as empirical models. First of all the non-Gaussian stable laws always have infinite variance. This makes them rather suspect as a realistic tool and prompts the introduction of *truncated* stable distributions which, however, are no longer stable. Then the range of the x decay rates of the probability density functions cannot exceed x^{-3}, and this too introduces a particular rigidity in these models. On the other hand the more general Lévy processes are generated by infinitely divisible laws and do not necessarily have these problems, but they can be more difficult to analyse and to simulate. Beside the fact that they do not have natural scaling properties, the laws of their increments could be explicitly known only at one time scale. In fact their time evolution is always given in terms of characteristic functions, but the marginal densities may not be calculable. This is a feature, however, that they share with most stable processes, since the probability density functions of the non-Gaussian stable laws are explicitly known only in precious few cases.

The need to go beyond the processes generated by stable distributions stems also from other recent advances in the field of the fractional differential equations. The evolution equations of the Lévy processes can be put in terms of pseudo-differential operators whose symbols are just the characteristic exponents of the processes (Jacob and Schilling 2001, Cont and Tankov 2004). The most popular form taken by these equations is that of the fractional differential equations, and this generalization of the diffusion equations can be put in connection with Lévy noises with non-Gaussian stable distributions (Gorenflo and Mainardi 1998a, 1998b, Metzler and Klafter 2000). It has been put in evidence in a few papers (Chechkin *et al* 2003, 2004), however, that in the case of Lévy flights confined by symmetric quartic potentials the stationary probability density functions show two unexpected properties: in fact not only they are bimodal, but they also have a finite variance, differently from what happens to the non-Gaussian, stable law of the system noise. This suggests that, under particular dynamical conditions, the stochastic evolutions produced by stable Lévy noises end up in nonstable distributions, and hence hints to a new physical interest beyond the pale of the stable laws.

Some new applications for the Lévy, infinitely divisible but not stable processes begin also to emerge in other physical domains (Cufaro Petroni *et al* 2005, 2006, Vivoli *et al* 2006): as we will see in the following the statistical characteristics of some recent model of the collective motion in the charged particle accelerator beams seem to point exactly in the direction of some kind of Student infinitely divisible process. At the present stage of our inquiry the proposed model for the particle beams is only phenomenological and it lacks a complete, underlying, physical mechanism producing the noise. This however brings to the fore the problem of the dynamical description of complex systems. The infinitely divisible Lévy processes with a jump component are indeed interesting also in the light of the connection established between Markov processes and quantum phenomena by the stochastic mechanics. This latter is a model universally known for its original application to the problem of building a classical stochastic model for quantum mechanics (Nelson 1967, 1985, Guerra 1981, Morato 1982, Guerra and Morato 1983), but in fact it is a very general model which is suitable for a large number of stochastic dynamical systems (Albeverio *et al* 1983, Paul and Baschnagel 1999, Cufaro Petroni *et al* 1999, 2000, 2003, 2004). As recently proposed, a stochastic mechanics with jumps driven by a non-Gaussian Lévy process could find applications in the physical and technological domain (Cufaro Petroni *et al* 2005, 2006). The presence of jumps could for instance be instrumental in building reasonable models for the formation of halos in beams of charged particles in accelerators. On the other hand this would not be the first time that Lévy processes find applications in quantum theory since they have already been used to build models for spinning particles (De Angelis and Jona-Lasinio 1982), for relativistic quantum mechanics (De Angelis 1990), and in stochastic quantization (Albeverio *et al* 2001).

The standard way to build a stochastic dynamical system is to modify the phase space dynamics by adding a Wiener noise $\mathbf{B}(t)$ to the momentum equation only, so that the usual relations between position and velocity are preserved:

$$m\,\mathrm{d}\mathbf{Q}(t) = \mathbf{P}(t)\,\mathrm{d}t, \qquad \mathrm{d}\mathbf{P}(t) = \mathbf{F}(t)\,\mathrm{d}t + \beta\,\mathrm{d}\mathbf{B}(t).$$

In this way we get a derivable, but non-Markovian position process $\mathbf{Q}(t)$. An example of this approach is that of a Brownian motion in a fluid described by an Ornstein-Uhlenbeck system of stochastic differential equations. Alternatively we can add a Wiener noise $\mathbf{W}(t)$ with diffusion coefficient D directly to the position equation:

$$\mathrm{d}\mathbf{Q}(t) = \mathbf{v}_{(+)}(\mathbf{Q}(t), t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}\mathbf{W}(t)$$

and get a Markovian, but not derivable $\mathbf{Q}(t)$. In this way the stochastic system is reduced to a single stochastic differential equation since we are obliged to drop the second (momentum) equation. The standard example of this reduction is the Smoluchowski approximation of the Ornstein-Uhlenbeck process in the overdamped case. As a consequence we will now work only in a configuration, and not in a phase space; but this does not prevent us from introducing a dynamics either by generalizing the Newton equations (Nelson 1967, 1985, Guerra 1981), or by means of a stochastic variational principle (Guerra and Morato 1983). From this stochastic dynamics, which now notably enjoys a measure of time-reversal invariance, two coupled equations can be derived which are equivalent to a Schrödinger equation, prompting the idea of a stochastic foundation of quantum mechanics. In fact the stochastic mechanics can be used to describe more general stochastic dynamical systems satisfying fairly general conditions: it is known since longtime (Morato 1982), for example, that for any given diffusion there is a correspondence between diffusion processes and solutions of this kind of Schrödinger equations where the Hamiltonians come from suitable vector potentials. The usual Schrödinger equation, and hence true quantum mechanics, is recovered when the diffusion coefficient coincides with $\hbar/2m$, namely is connected to the Planck constant. However we are interested here not only in a stochastic model of quantum mechanics, but also to the general description of complex systems as a particle beams, and to this end it would be very interesting—as already remarked—to be able to generalize the stochastic mechanical scheme to the case of non-Gaussian Lévy noises. The road to this end, however, is fraught with technical difficulties, so that a better understanding of the possible underlying Lévy noises should be considered as a first, unavoidable step.

In this light the aim of this paper is to study a few examples of nonstable, infinitely divisible processes, and in particular we will focus our attention on the Student processes. Since the Student family of laws is infinitely divisible but nonclosed under convolution the process distribution will not be Student at every time. We will show however that, at least in particular cases, the process transition law is a mixture of a finite number of Student laws, and it is suggested that this could be a general feature of the Student processes. On the other hand it can also be seen that for every finite time the spatial asymptotic behaviour always is the same as that of the Student distribution at the characteristic time scale; and this turns out to be exactly the behaviour put in evidence by Vivoli *et al* (2006) in the solutions of the complex dynamical system used to study the behaviour of beams of charged particles in accelerators.

We will limit our considerations to one-dimensional models without going into the problem of the dependence structure of a multivariate process (see for example Cont and Tankov 2004), and we will not pretend any completeness or generality: our aim is rather to present the features of a few selected processes to gain a deeper insight into their possible general behaviour. The paper is organized as follows: in section 2 we recall a few, well-known facts about the Lévy processes and in particular the connection between the transition function $p(x,t|y,s)$ and a triplet of functions $A(y,s)$, $B(y,s)$ and $W(x|y,s)$ characteristic of a Lévy process. We also propose a different simplified, heuristic procedure to find the explicit form of A, B and W: a procedure not completely general, but which works well enough for the rather regular transition functions discussed in this paper. In the section 3 we analyse the behaviour of two families of laws (the variance gamma, and the Student laws) which are particular limit cases of a larger class of infinitely divisible laws: that of the generalized hyperbolic laws which received considerable attention in recent years (Raible (2000), Eberlein and Raible (2000), Eberlein (2001), Cont and Tankov (2004), and references quoted therein). Our two families are in a certain sense conjugate to each other since the roles of their probability density functions and characteristic functions are interchanged. Let us remark here that all the laws that we take in consideration in this paper are infinitely divisible, but—with a few notable exceptions—not stable. We then pass in section 4 to study the Lévy process produced by the variance gamma distributions: since this class is closed under convolution, it will be easy enough to find both the characteristic triplet, and the laws of the increments for every value of the time interval. Apparent similarities notwithstanding the case of the Student processes discussed in section 5 are rather different from the previous one. In fact the Student family is not even closed under convolution so that we do not have explicit expressions for the transition laws at every time scale. As a consequence we will restrict our attention to a subclass of Student processes by choosing particular (but not trivial) values for the parameters, and we will get results about (a) the spatial asymptotic behaviour of the transition functions at every time, (b) the explicit form of the transition functions at time intervals which are integral multiples of a characteristic time constant and (c) the form of the Lévy triplet of functions. In particular we will find that, at discrete times, the process law turns out to be a mixture of a finite number of Student distributions by means of a new kind of time-dependent discrete probability distribution. We finally discuss in section 6 some pathwise properties of our nonstable processes by showing also a few simulations of the Ornstein-Uhlenbeck processes driven by our Lévy noises, and conclude with some remark about the perspectives of future research.

2. Lévy processes generated by *id* laws

A Lévy process $X(t)$ is a stationary, stochastically continuous, independent increment Markov process. It is well known that the simplest way to produce its transition laws is to start with a *type* of infinitely divisible (*id*) distributions (see, Gnedenko and Kolmogorov (1968), Loève (1978), (1987) and Sato (1999) for a more recent monograph): if we focus our attention on centred laws, a type of these *generating* laws can be given by the family of their characteristic functions (*chf*) $\varphi(au)$ with a spatial scale parameter $a > 0$. The *chf* of the transition law of our stationary process in the time interval $[s,t]$ will then be

$$\Phi(au, t-s) = [\varphi(au)]^{(t-s)/T}, \tag{1}$$

where T is a suitable constant playing the role of a time scale parameter, while the transition probability density function (*pdf*) with initial condition $X(s) = y$, $\mathbb{P}$-q.o. will be recovered by an inverse Fourier transform

$$\begin{aligned} p(x,t|y,s) &= \frac{1}{2\pi}\lim_{M\to+\infty}\int_{-M}^{M}\Phi(au,t-s)\,\mathrm{e}^{-\mathrm{i}(x-y)u}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\lim_{M\to+\infty}\int_{-M}^{M}[\varphi(au)]^{(t-s)/T}\,\mathrm{e}^{-\mathrm{i}(x-y)u}\,\mathrm{d}u \end{aligned} \tag{2}$$

and—because of stationarity—will only depend on the differences $x - y$ and $t - s$.

The parameters a and T play a role in the *scale invariance* properties of the process. When the generating family of *id* laws is closed under convolution the transition laws remain within this same family all along the evolution, and the changes are summarized just in a time dependence of some parameter of the *pdf*. But in the case of *stable* laws there is more. If for instance—as in the Wiener process—the generating type of law is the normal, centred $\mathcal{N}(0, a)$ it is well known that the transition law (with $y = 0$ and $s = 0$ for simplicity) is just $\mathcal{N}(0, a\sqrt{t/T})$, namely it is always normal, but with a time-dependent parameter: the variance, changing linearly with the time as Dt, where $D = a^2/T$ is the diffusion coefficient. This means that the overall behaviour of the process is ruled only by D, and not by a and T separately. As a consequence the particular values of a and T, namely the particular units of measurement, are immaterial and we have the scale invariance. This gives to the Wiener process its property of *self-similarity*: no matter at what space-time scale (namely irrespectively to the values you give to a and T, provided that $D = a^2/T$ keeps the same value) you choose to observe the process, the trajectories always will look the same.

These properties of the Wiener process are shared by all the other Lévy processes generated by stable—even non-normal—laws, but not in general by the processes generated by other, nonstable *id* laws. It must be remarked, however, that all the non-Gaussian stable laws do not have a finite variance, and show a rather restricted range of possible decays for large x: features that partly impair a realistic use of them in empirical situations. On the other hand families of nonstable, *id* laws can still be closed under convolution, as it is for instance the case of the compound Poisson laws $\mathcal{P}(\lambda, a; \chi)$ with *chf* $\varphi(au) = \mathrm{e}^{\lambda[\chi(au)-1]}$, where $\chi(u)$ is the *chf* of the jump distribution. This means again that the evolution of the transition law of a compound Poisson process can always be summarized in the time dependence of the Poisson parameter as $\mathcal{P}(\lambda t/T, a; \chi)$, but with respect to the Wiener case there are important differences: while all the transition laws of a Wiener process belong to the same (normal) type, Poisson transition laws with different parameters do not. The normal laws are indeed stable, while the Poisson laws are only *id*, and Poisson laws with different values of λ do not belong to the same type. Moreover, while a change in the T value can always be compensated by a corresponding change of λ so that λ/T remains the same, the roles of a and T in a compound Poisson process, at variance with the Wiener case, remain completely separated and we do not have the same kind of self-similarity.

The less simple case of processes is finally that generated by families of *id* laws which are not even closed under convolution, since in this event the transition distributions do not remain within the same family, and the overall evolution cannot be summarized just in the time dependence of some parameter. As we will see in the following this is far to be an uncommon situation and this paper is mainly devoted to the analysis of particular processes of this kind. It must be kept in mind that in this last case the role of the scale parameters becomes relevant since a change in their values can no longer be compensated by reciprocal changes in other parameters. This means that, to a certain extent, a change in these scale constants produces different processes, so that for instance we are no longer free to look at the process at different time scales by presuming to see the same features. We should remark, on the other hand, that—at variance with the stable, non-Gaussian case—the *pdf*'s of the *id* distributions can have both a wide range of decay laws for $|x| \to +\infty$, and a finite variance σ^2. For these Lévy processes generated by *id* laws with finite variance σ^2 it is finally easy to see that—due to the fact that the process has independent increments—the variance always is finite and grows linearly with the time as $\sigma^2 t/T$: a feature typical of the ordinary (non-anomalous) diffusions.

2.1. The decomposition of a Lévy process

The evolution equations of a process driven by a Lévy noise can be given either as partial integro-differential equations (PIDE) for the transition functions of the process (Loève 1978, Gardiner 1997), or as stochastic differential equations (SDE) for its trajectories (Applebaum 2004, Øksendal and Sulem 2005, Protter 2005). In both cases the structure of the evolution is given in terms of some characteristic triplet of functions. For simple Lévy process of course this triplet will give rise just to its Lévy decomposition in a drift, a Brownian and a jump term. In this paper we will choose to follow the description in terms of PIDE, and it is important to recall how this characteristic triplet is related to the transition functions. We will not attempt to give here a complete and rigorous survey of the argument, but we will limit ourselves to fix the notation in a rather simplified form (see for example, Gardiner (1997), but also for a more rigorous approach Léandre (1987), Ishikawa (1994), Sato (1999), Barndorff-Nielsen (2000) and Rüschendorf and Woerner (2002)) suitable for the cases that we will analyse. In particular we suppose to consider only processes endowed with well behaved *pdf*'s, so that (apart from an initial distribution) the process is completely defined by its transition *pdf* $p(x, t|y, s)$. If then we define the triplet of functions

$$A(y, s) = \lim_{\epsilon\to 0^+} \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|x-y|<\epsilon} (x - y) p(x, s + \Delta t|y, s)\, \mathrm{d}x \qquad (3)$$

$$B(y, s) = \lim_{\epsilon\to 0^+} \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|x-y|<\epsilon} (x - y)^2 p(x, s + \Delta t|y, s)\, \mathrm{d}x \qquad (4)$$

$$W(x|y, s) = \lim_{\Delta t\to 0} \frac{p(x, s + \Delta t|y, s)}{\Delta t}, \qquad x \neq y \qquad (5)$$

it can be seen that the *pdf*'s of the process satisfy the following (*forward*) PIDE

$$\partial_t p(x, t) = -\partial_x[A(x, t)p(x, t)] + \frac{1}{2}\partial_x^2[B(x, t)p(x, t)] + \lim_{\epsilon\to 0^+} \int_{|x-z|\geqslant\epsilon} [W(x|z, t)p(z, t) - W(z|x, t)p(x, t)]\, \mathrm{d}z \qquad (6)$$

the transition *pdf* being the solution corresponding to the initial condition $p(x, s^+|y, s) = \delta(x - y)$. In the case of stationary processes (as our Lévy processes are) the transition *pdf* $p(x, t|y, s)$ depends on its variables only through their differences $x - y$ and $t - s$. As a consequence A and B are simply constants, while $W(x|y, s) = W(x - y)$. It is also known that A plays the role of a drift coefficient, while B is a diffusion coefficient connected to the Brownian component of the process; finally $W(x|y, s)$, defined only for $x \neq y$, is the density of the Lévy measure of the process. The knowledge of the characteristic triplet is also instrumental to write down the PIDE (or alternatively the SDE) for other processes driven by a Lévy noise.

In order to calculate the characteristic triplet of a Lévy process decomposition from (3), (4) and (5) we are supposed to explicitly know its transition *pdf*. We will see in the following, however, that given the *chf*'s of an *id* distribution it is very easy to write the *chf* (1) of the process increments, but also that in general it is not a simple task to explicitly calculate the transition *pdf* by the inverse Fourier transform (2). We then propose here a different procedure to calculate A, B and W directly from the process *chf* which is surely a known quantity for a Lévy process, by adding however that at the present stage its derivation is only heuristic. To this end let us remark that from (1) and (2) the transition *pdf* will have the form

$$p(x, s + \Delta t|y, s) = \frac{1}{2\pi} \lim_{M\to +\infty} \int_{-M}^{M} [\varphi(au)]^{\Delta t/T}\, \mathrm{e}^{-\mathrm{i}u(x-y)}\, \mathrm{d}u$$

so that, by supposing (which is fair for all the cases that we will consider in this paper) $\varphi(-\infty) = \varphi(+\infty) = 0$, we get with an integration by parts

$$\frac{p(x, s + \Delta t | y, s)}{\Delta t} = \frac{a}{2\pi \mathrm{i}(x - y)T} \lim_{M\to+\infty} \int_{-M}^{M} [\varphi(au)]^{\Delta t/T} \frac{\varphi'(au)}{\varphi(au)} \,\mathrm{e}^{-\mathrm{i}u(x-y)} \,\mathrm{d}u.$$

If now we suppose that our functions are regular enough to allow both to exchange the two limits for $\Delta t \to 0$ and for $M \to +\infty$, and to perform the limit for $\Delta t \to 0$ under the integral, we immediately have

$$W(x|y, s) = W(x - y) = \frac{a}{2\pi \mathrm{i}(x - y)T} \lim_{M\to+\infty} \int_{-M}^{M} \frac{\varphi'(au)}{\varphi(au)} \,\mathrm{e}^{-\mathrm{i}u(x-y)} \,\mathrm{d}u$$

namely with $z = x - y$

$$W(z) = \frac{a}{2\pi \mathrm{i}zT} \lim_{M\to+\infty} \int_{-M}^{M} \frac{\varphi'(au)}{\varphi(au)} \,\mathrm{e}^{-\mathrm{i}uz} \,\mathrm{d}u. \tag{7}$$

Remark that in (7) the limit must be understood in the sense of the distributions, as can be easily checked by applying the formula to some well-known case (either the Wiener, or the Cauchy process). What is most interesting with respect to equation (5) is that now we can calculate $W(z)$ directly from $\varphi(au)$, without explicitly knowing the transition *pdf* $p(x, t|y, s)$.

In the same way for A with an integration by parts we have first of all that

$$\frac{1}{\Delta t} \int_{|x-y|<\epsilon} (x - y) p(x, s + \Delta t|y, s) \,\mathrm{d}x$$
$$= \frac{a}{2\pi \mathrm{i}T} \int_{|x-y|<\epsilon} \left[\lim_{M\to+\infty} \int_{-M}^{M} [\varphi(au)]^{\Delta t/T} \frac{\varphi'(au)}{\varphi(au)} \,\mathrm{e}^{-\mathrm{i}u(x-y)} \,\mathrm{d}u \right] \mathrm{d}x$$

then, if again it is allowed to freely exchange limits and integrals, we have

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|x-y|<\epsilon} (x - y) p(x, s + \Delta t|y, s) \,\mathrm{d}x$$
$$= \frac{a}{2\pi \mathrm{i}T} \int_{|x-y|<\epsilon} \left[\lim_{M\to+\infty} \int_{-M}^{M} \frac{\varphi'(au)}{\varphi(au)} \,\mathrm{e}^{-\mathrm{i}u(x-y)} \,\mathrm{d}u \right] \mathrm{d}x$$
$$= \frac{a}{\mathrm{i}\pi T} \lim_{M\to+\infty} \int_{-M}^{M} \frac{\varphi'(au)}{\varphi(au)} \frac{\sin u\epsilon}{u} \,\mathrm{d}u$$

and finally

$$A(y, s) = A = \frac{a}{\mathrm{i}\pi T} \lim_{\epsilon\to 0^+} \lim_{M\to+\infty} \int_{-M}^{M} \frac{\varphi'(au)}{\varphi(au)} \frac{\sin u\epsilon}{u} \,\mathrm{d}u. \tag{8}$$

Here it is understood that the two limits (always in the sense of distributions) and the integration must be performed in the order indicated since an exchange will produce a trivial—and wrong—result. Remark that when $\varphi(au)$ is an even function (as happens if the process increments are symmetrically distributed around zero), then $\varphi'(au)/\varphi(au)$ is an odd function, and hence—since $u^{-1} \sin u\epsilon$ is even—we immediately get $A = 0$. This is coherent with the fact that, when the increments are symmetrically distributed, then we do not expect to have a drift in the process.

As for the coefficient B the usual integration by parts gives

$$\frac{1}{\Delta t} \int_{|x-y|<\epsilon} (x - y)^2 p(x, s + \Delta t|y, s) \,\mathrm{d}x$$
$$= \frac{a}{2\pi \mathrm{i}T} \int_{|x-y|<\epsilon} \left[\lim_{M\to+\infty} \int_{-M}^{M} [\varphi(au)]^{\Delta t/T} \frac{\varphi'(au)}{\varphi(au)} (x - y) \,\mathrm{e}^{-\mathrm{i}u(x-y)} \,\mathrm{d}u \right] \mathrm{d}x$$

so that by exchanging limits and integrals we get

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-y|<\epsilon}(x-y)^2 p(x,s+\Delta t|y,s)\,\mathrm{d}x$$
$$=\frac{a}{2\pi\mathrm{i}T}\int_{|x-y|<\epsilon}\left[\lim_{M\to+\infty}\int_{-M}^{M}\frac{\varphi'(au)}{\varphi(au)}(x-y)\,\mathrm{e}^{-\mathrm{i}u(x-y)}\,\mathrm{d}u\right]\mathrm{d}x$$
$$=\frac{a}{\pi T}\lim_{M\to+\infty}\int_{-M}^{M}\frac{\varphi'(au)}{\varphi(au)}\frac{u\epsilon\cos u\epsilon-\sin u\epsilon}{u^2}\,\mathrm{d}u$$

and finally our coefficient is

$$B(y,s)=B=\frac{a}{\pi T}\lim_{\epsilon\to 0^+}\lim_{M\to+\infty}\int_{-M}^{M}\frac{\varphi'(au)}{\varphi(au)}\frac{u\epsilon\cos u\epsilon-\sin u\epsilon}{u^2}\,\mathrm{d}u. \tag{9}$$

Also in this case we see that for our stationary, independent increment process this coefficient is a constant independent from the initial coordinates y and s.

The formulae (7), (8) and (9) can finally be checked on two well-known (stable) cases to give the correct characteristic triplets: the Wiener process produced by a normal distribution $\mathcal{N}(0,a)$ with

$$A=0,\qquad B=\frac{a^2}{T},\qquad W(z)=0 \tag{10}$$

and the Cauchy process produced by a Cauchy distribution $\mathcal{C}(a)$ with

$$A=0,\qquad B=0,\qquad W(z)=\frac{a}{\pi T z^2}. \tag{11}$$

Remark as in these two stable cases the elements of the triplet do not depend separately on the two (time and space) scale parameters, but only on a combination of them so that a change in the time scale can always be compensated by an exchange in the space scale (and vice versa): a point giving rise to the scale invariance which in general is not reproduced in nonstable processes, as discussed at the beginning of this section.

3. A class of infinitely divisible distributions

The increment laws of the Lévy processes analysed in this paper are particular (limiting) cases of a larger class of distributions, that of the generalized hyperbolic (GH) distributions (for their general properties see for example, Raible (2000), Eberlein and Raible (2000), Eberlein (2001), Cont and Tankov (2004), and references quoted therein). The GH distributions constitute a five-parameter class of *id*, absolutely continuous laws with the following *pdf*'s (for $x\in\mathbb{R}$) and *chf*'s

$$f(x+\mu)=\frac{\mathrm{e}^{\beta x}}{\alpha^{2\lambda-1}\delta^{2\lambda}\sqrt{2\pi}}\frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}(\alpha\sqrt{\delta^2+x^2})^{\lambda-\frac{1}{2}}K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+x^2})$$
$$\varphi(u)=\mathrm{e}^{\mathrm{i}\mu u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2})}{(\delta\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2})^\lambda},$$

where $\lambda\in\mathbb{R}$, $\alpha>0$, $\beta\in(-\alpha,\alpha)$, $\delta>0$, $\mu\in\mathbb{R}$, and $K_\nu(z)$ are the modified Bessel functions (Abramowitz and Stegun 1968). Apparently α and δ play the role of scale parameters, while β is a skewness parameter: the *pdf* is symmetric when $\beta=0$. On the other hand μ is just a centring parameter: since in this paper our attention will be focused on the symmetric, centred laws, we will always choose $\beta=0$ and $\mu=0$ and we will consider the more restricted (but still large enough) class $\mathcal{GH}(\lambda, \alpha, \delta)$ of the centred, symmetric GH laws with the following *pdf*'s and *chf*'s:

$$f_{\mathrm{GH}}(x) = \frac{\alpha}{(\delta\alpha)^\lambda K_\lambda(\delta\alpha)\sqrt{2\pi}} (\alpha\sqrt{\delta^2 + x^2})^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}(\alpha\sqrt{\delta^2 + x^2}) \tag{12}$$

$$\varphi_{\mathrm{GH}}(u) = \frac{(\delta\alpha)^\lambda}{K_\lambda(\delta\alpha)} \frac{K_\lambda(\delta\sqrt{\alpha^2 + u^2})}{(\delta\sqrt{\alpha^2 + u^2})^\lambda} \tag{13}$$

with $\lambda \in \mathbb{R}$, $\alpha > 0$, and $\delta > 0$.

The GH class contains many relevant particular cases, also for limit values of the parameters, and its name comes from the fact that it contains as sub-class with $\lambda = 1$ that of the hyperbolic distributions called in this way because the logarithm of their *pdf* is a hyperbola. The GH distributions are not always endowed with finite momenta: this fact depends on the parameter values and must be explicitly assessed for every particular case. On the other hand they are all *id*, and hence they are good starting points to build Lévy processes. In general, however, they are not stable laws, and in fact they are not even closed under convolution: the sum of two GH random variables (rv) is not a GH rv. This means not only that the corresponding processes will not be self-similar, but also that often it is not easy to find out what the *pdf* of the process looks like even if it is well known at one time. Remark that the GH class is rich enough to contain also as a limit case the sub-class of the normal laws $\mathcal{N}(\mu, \sigma)$. Indeed it can be shown that (in distribution)

$$\lim_{\delta \to +\infty} \lim_{\lambda \to -\infty} \lim_{\alpha \to 0^+} \mathcal{GH}(\lambda, \alpha, \delta) = \mathcal{N}(0, \sigma)$$

provided that $\delta^2/|\lambda| \to 2\sigma^2$. In the following we will study the behaviour of the processes produced by two other particular limit sub-classes that, at variance with the normal distributions, are not stable besides a few exceptions.

3.1. The variance gamma distributions

The variance gamma (VG) laws (Madan and Seneta 1987, 1990, Madan and Milne 1991, Madan *et al* 1998) are obtained from $\mathcal{GH}(\lambda, \alpha, \delta)$ in the limit for $\delta \to 0^+$. More precisely, since in general

$$K_\nu(z) = K_{-\nu}(z) \sim \begin{cases} \frac{1}{2}\Gamma(\nu)(2/z)^\nu, & \text{for} \quad \nu > 0, \\ -\log z, & \text{for} \quad \nu = 0, \\ \frac{1}{2}\Gamma(|\nu|)(2/z)^{|\nu|}, & \text{for} \quad \nu < 0, \end{cases} \qquad z \to 0 \tag{14}$$

we have for $\lambda > 0$ that

$$\lim_{\delta \to 0^+} (\delta\alpha)^\lambda K_\lambda(\delta\alpha) = 2^{\lambda - 1}\Gamma(\lambda)$$

and hence the *pdf*'s of the centred, symmetric VG laws—which constitute the two parameters family $\mathcal{VG}(\lambda, \alpha)$—are

$$f_{\mathrm{VG}}(x) = \frac{2\alpha}{2^\lambda \Gamma(\lambda)\sqrt{2\pi}} (\alpha|x|)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}(\alpha|x|). \tag{15}$$

with $\lambda > 0$ and $\alpha > 0$. As for the corresponding *chf*'s of $\mathcal{VG}(\lambda, \alpha)$ it is readily seen from (13) and (14) that they simply reduce to

$$\varphi_{\mathrm{VG}}(u) = \left(\frac{\alpha^2}{\alpha^2 + u^2}\right)^\lambda. \tag{16}$$

It is apparent that α plays the role of a scale parameter, while λ classifies the different types of VG laws. For $\lambda = 1$ the *pdf*'s and *chf*'s of the $\mathcal{VG}(1, \alpha)$ laws are

$$f(x) = \frac{\alpha}{2}\,\mathrm{e}^{-\alpha|x|}, \qquad \varphi(u) = \frac{\alpha^2}{\alpha^2 + u^2}$$

so that $\mathcal{VG}(1, \alpha)$ is nothing but the class of the Laplace (double exponential) laws $\mathcal{L}(\alpha)$. The *pdf* (15) has an elementary form only when $\lambda = n + 1$ is an integer with $n = 0, 1, \ldots$. In fact from

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\sum_{j=0}^{n}\frac{(n+j)!}{j!(n-j)!}\frac{1}{(2z)^j} \tag{17}$$

it is easy to see that (with $\ell = n - j$) we have

$$f_{\mathrm{VG}}(x) = \frac{\alpha}{2^{2n+1}}\,\mathrm{e}^{-\alpha|x|}\sum_{\ell=0}^{n}\binom{2n-\ell}{n}\frac{(2\alpha|x|)^\ell}{\ell!}, \qquad \lambda = n + 1 = 1, 2, \ldots$$

For $\lambda \to 0$ equation (16) shows also that our VG laws converge in law to a distribution degenerate in $x = 0$. From the asymptotic behaviour of the Bessel functions

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}, \qquad |z| \to +\infty \tag{18}$$

we immediately see that the asymptotic behaviour of the *pdf* (15) is $(\alpha|x|)^\lambda\,\mathrm{e}^{-\alpha|x|}$, and hence the momenta always exist for every $\lambda \in \mathbb{R}$. Of course this corresponds to the fact that the *chf* (16) is always derivable in $u = 0$. Since our laws are centred and symmetric the odd momenta vanish; as for the even momenta we have by direct calculation

$$m_{\mathrm{VG}}(2k) = \frac{2^k(2k-1)!!}{\alpha^{2k}}\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}, \qquad k = 0, 1, 2, \ldots$$

so that the expectation is always zero, and the variance

$$\sigma^2_{\mathrm{VG}} = \frac{2\lambda}{\alpha^2}.$$

Then it is easy to see that for a given $\sigma > 0$ the laws $\mathcal{VG}(\lambda, \sqrt{2\lambda}/\sigma)$ have all the same variance σ^2 for every value of λ, and that for $\lambda \to +\infty$ they converge in distribution to the normal law $\mathcal{N}(0, \sigma)$. From the *chf* (16) we immediately see that the VG distributions are *id* but not stable. It is easy to see, however, that, the sub-families $\mathcal{VG}(\lambda, \alpha)$ with a fixed value of α are closed under convolution: in fact the sum of two independent rv's respectively with laws $\mathcal{VG}(\lambda_1, \alpha)$ and $\mathcal{VG}(\lambda_2, \alpha)$ is a rv with law $\mathcal{VG}(\lambda_1 + \lambda_2, \alpha)$, as can easily be seen from (16). This of course does not amount to stability since laws $\mathcal{VG}(\lambda, \alpha)$ and $\mathcal{VG}(\lambda', \alpha)$ with $\lambda \neq \lambda'$ are not of the same type. For the sake of simplicity in the following we will take $\alpha = 1$ and we will use the shorthand notation $\mathcal{VG}(\lambda) = \mathcal{VG}(\lambda, 1)$.

3.2. The Student distributions

The class of the centred, symmetric Student laws (see Heyde and Leonenko (2005) for a recent review) can be considered as conjugate to that of the centred, symmetric VG laws in the sense that here the roles of the *pdf* and *chf* are interchanged. They are the limit for $\alpha \to 0^+$ of the $\mathcal{GH}(\lambda, \alpha, \delta)$ laws with $\lambda < 0$. By taking the new parameter $\nu = -2\lambda > 0$, and recalling that $K_\nu(z) = K_{-\nu}(z)$, the *pdf* and *chf* of the $\mathcal{GH}(\lambda, \alpha, \delta)$ laws become

$$f_{\mathrm{GH}}(x) = \frac{\alpha}{\sqrt{2\pi}}\frac{(\delta\alpha)^{\frac{\nu}{2}}}{K_{\frac{\nu}{2}}(\delta\alpha)}\frac{K_{\frac{\nu+1}{2}}(\alpha\sqrt{\delta^2+x^2})}{(\alpha\sqrt{\delta^2+x^2})^{\frac{\nu+1}{2}}}$$

$$\varphi_{\mathrm{GH}}(u) = \frac{(\delta\sqrt{\alpha^2+u^2})^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(\delta\sqrt{\alpha^2+u^2})}{(\delta\alpha)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(\delta\alpha)}$$

so that from equation (14) in the limit for $\alpha \to 0^+$ we get the *pdf* and *chf* of the centred, symmetric Student laws $\mathcal{T}(\nu, \delta)$

$$f_{\mathrm{ST}}(x) = \frac{1}{\delta B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(\frac{\delta^2}{\delta^2 + x^2}\right)^{\frac{\nu+1}{2}} \tag{19}$$

$$\varphi_{\mathrm{ST}}(u) = 2\frac{(\delta|u|)^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(\delta|u|)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}, \tag{20}$$

where $\nu > 0$, $\delta > 0$ and $B(z, w)$ is the Beta function (Abramowitz and Stegun 1968). Here δ is the scale parameter, while ν classifies the different law types. It is also easy to see that for $|x| \to +\infty$ the Student *pdf* goes to zero as $|x|^{-\nu-1}$, so that for a given ν the moments $m_{\mathrm{ST}}(n)$ exist only if $n < \nu$. When they exist, the odd momenta are zero for symmetry, while the even momenta are

$$m_{\mathrm{ST}}(2k) = \delta^{2k} \frac{B\left(\frac{1}{2}+k, \frac{\nu}{2}-k\right)}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)}, \qquad k = 0, 1, 2, \ldots, \quad 2k < \nu.$$

In particular the expectation exists (and vanishes) for $\nu > 1$, while the variance exists finite for $\nu > 2$ and its value is

$$\sigma_{\mathrm{ST}}^2 = \frac{\delta^2}{\nu - 2}. \tag{21}$$

As a consequence, for $\nu > 2$ and for a given $\sigma > 0$, the laws $\mathcal{T}(\nu, \sigma\sqrt{\nu-2})$ have all the same variance σ^2, and it is easy to show that for $\nu \to +\infty$ they converge in distribution to the normal law $\mathcal{N}(0, \sigma)$. It can be proved that the Student distributions are *id* (this is not trivial at all; see Grosswald (1976a, 1976b), Ismail (1977), Bondesson (1979, 1992), Pitman and Yor (1981)), but that they are not stable, with one notable exception: the $\nu = 1$ case, that of the Cauchy laws $\mathcal{T}(1, \delta) = \mathcal{C}(\delta)$ which constitute one of the better known classes of stable laws with *pdf* and *chf*

$$f(x) = \frac{1}{\delta\pi} \frac{\delta^2}{\delta^2 + x^2}, \qquad \varphi(u) = \mathrm{e}^{-\delta|u|}.$$

Besides this case—and at variance with the VG—the Student laws are not even closed under convolution: this makes the study of the time evolution of a Student process a more complicated and interesting business which constitutes a relevant part of this paper. For the sake of simplicity in the following we will take $\delta = 1$ and we will use the shorthand notation $\mathcal{T}(\nu) = \mathcal{T}(\nu, 1)$.

4. The VG process

Lévy processes produced by means of VG distributions are simple enough because of their closure under convolution. In fact it is easy to see from (16) that (taking $\alpha = 1$ and $T = 1$ to simplify the notations) for a $\mathcal{VG}(\lambda)$ law the transition *chf* of the process (with initial time $s = 0$ and position $y = 0$) is

$$\Phi(u, t|\lambda) = [\varphi_{\mathrm{VG}}(u)]^t = \left(\frac{1}{1+u^2}\right)^{\lambda t} \tag{22}$$

so that the law of the increment in $[0, t]$ always is a VG law with the parameter evolving in time; namely, at every t, we have $X(t) \sim \mathcal{VG}(\lambda t)$, and hence the corresponding *pdf* is explicitly known at every time and is

$$p(x, t|\lambda) = \frac{2}{2^{\lambda t}\Gamma(\lambda t)\sqrt{2\pi}}|x|^{\lambda t - \frac{1}{2}} K_{\lambda t - \frac{1}{2}}(|x|). \tag{23}$$

Apparently—as in the Poisson case—the laws of the process belong to the VG family all along the evolution, but this does not mean that the process is stable since the laws of the VG family are not of the same type. In fact, with increasing values of t, the distributions of a VG process go throughout all the gamut of the VG family: what changes with λ is just the instant when the distribution is simply a bilateral exponential. As remarked in the section 3.1 the *pdf* (23) has an elementary form only for $t = \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots$ but a great deal of information is available also in the general, nonelementary form. In particular from (14) and (18) we can study the behaviour of the *pdf* both near the origin and in the asymptotic region. For small x we find

$$p(x, t|\lambda) \sim \begin{cases} |x|^{2\lambda t - 1}, & \text{for} \quad 0 < t < \frac{1}{2\lambda}, \\ -\log|x|, & \text{for} \quad t = \frac{1}{2\lambda}, \\ \frac{1}{2\pi}\frac{\Gamma(\lambda t - \frac{1}{2})}{\Gamma(\lambda t)}, & \text{for} \quad \frac{1}{2\lambda} < t, \end{cases} \qquad x \to 0$$

namely near the origin the *pdf* has an integrable singularity for $0 < t \leqslant \frac{1}{2\lambda}$, and thereafter it takes finite values for $t > \frac{1}{2\lambda}$. As for the asymptotic behaviour we have

$$p(x, t|\lambda) \sim |x|^{\lambda t - 1}\, \mathrm{e}^{-|x|}, \qquad |x| \to +\infty$$

namely it is a negative exponential times a power. It is apparent then that this asymptotic behaviour changes with time since the power depends on t; it is however always dominated by the exponential so that all the moments exist at every time. From equation (23) we can also explicitly calculate the characteristic triplet:

$$A = 0, \qquad B = 0, \qquad W(z) = \lambda\frac{\mathrm{e}^{-|z|}}{|z|} \tag{24}$$

so that the dimensionless PIDE for the VG process takes the form

$$\partial_t p(x, t) = \lim_{\epsilon \to 0^+} \int_{|z| \geqslant \epsilon} \lambda\, \mathrm{e}^{-|z|} \frac{p(x+z, t) - p(x, t)}{|z|}\, \mathrm{d}z.$$

A validation of (24) comes then from the Lévy–Khinchin formula (Loève 1987) which here reads

$$\log\varphi(u) = \lim_{\epsilon \to 0^+} \int_{|x| \geqslant \epsilon} \left(\mathrm{e}^{\mathrm{i}ux} - 1 - \frac{\mathrm{i}ux}{1 + x^2}\right) W(x)\, \mathrm{d}x \tag{25}$$

and which from (16) and (24) easily reduces itself to

$$-\log(1 + u^2) = 2\int_0^{+\infty} (\cos ux - 1)\frac{\mathrm{e}^{-x}}{x}\, \mathrm{d}x$$

a relation which is immediately verified by direct calculation.

5. The Student process

Despite their apparent symmetry and analogy with the VG family, the processes produced by the Student laws are not so straightforward to analyse (for recent results about the Student process see Heyde and Leonenko (2005)). The problem is that the Student family $\mathcal{T}(\nu, \delta)$ is not even closed under convolution, so that it is not easy to figure out the general behaviour of a Student process with arbitrary ν. As a consequence we will limit ourselves here to study the particular case of the $\nu = 3$ process whose features can be fairly understood: this will also give us an insight on the possible general behaviour of these Lévy processes. It is important to remark, moreover, that this particular Student process with $\nu = 3$ is the present candidate to describe the increments in the velocity process for particles in an accelerator beam (Vivoli *et al* 2006), and hence its analysis has not a purely academic interest. Let us introduce now the following notation for the $\mathcal{T}(\nu, \delta)$ laws and the corresponding processes: for $\nu > 0$ and $\delta > 0$

$$f(x|\nu, \delta) = f_{\mathrm{ST}}(x) = \frac{1}{\delta B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(\frac{\delta^2}{\delta^2 + x^2}\right)^{\frac{\nu+1}{2}} \tag{26}$$

$$f(x|\nu) = f(x|\nu, 1) = \frac{1}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \left(\frac{1}{1 + x^2}\right)^{\frac{\nu+1}{2}} \tag{27}$$

so that $f(x|\nu)$ from now on will be the *pdf* of the Student law $\mathcal{T}(\nu) = \mathcal{T}(\nu, 1)$ for $\delta = 1$. In the same way we can introduce the reduced form of the *chf*

$$\varphi(u|\nu, \delta) = \varphi_{\mathrm{ST}}(u) = 2\frac{|\delta u|^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(|\delta u|)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}$$

$$\varphi(u|\nu) = \varphi(u|\nu, 1) = 2\frac{|u|^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(|u|)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}.$$

Then, by taking $T = 1$, the transition *chf* of the Student process for the law $\mathcal{T}(\nu)$ (with initial time $s = 0$ and position $y = 0$) is explicitly known and is

$$\Phi(u, t|\nu) = [\varphi(u|\nu)]^t$$

and the corresponding transition *pdf* is

$$p(x, t|\nu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ux} \Phi(u, t|\nu) \, \mathrm{d}u = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ux} [\varphi(u|\nu)]^t \, \mathrm{d}u.$$

If we denote as $\mathcal{T}(\nu, \delta)$-process the Student process such that its law at $t = T$ is exactly $\mathcal{T}(\nu, \delta)$ then $p(x, t|\nu)$ will be the *pdf* of a $\mathcal{T}(\nu)$-process. In the following we will perform our calculations on the reduced, dimensionless quantities only: we can always revert to the dimensional variables by means of simple transformations. It is easy to realize from the form of $\Phi(u, t|\nu)$ that for $t \to 0^+$ the process approaches a law degenerate in $x = 0$, and that along the evolution of a Student process the marginal $p(x, t|\nu)$ no longer are simple Student *pdf*'s: after all we know that the Student family is neither stable, nor closed under convolution. The main problem is then to find an explicit form for the transition *pdf* which by symmetry can be explicitly written as

$$p(x, t|\nu) = \frac{1}{\pi} \int_0^{+\infty} \cos(ux) \left[2\frac{|u|^{\frac{\nu}{2}} K_{\frac{\nu}{2}}(|u|)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}\right]^t \mathrm{d}u. \tag{28}$$

5.1. The Student processes of odd integer index: the $\nu = 3$ case

Since the integration in (28) cannot be performed in general we will limit ourselves to particular cases. To do that let us remark that the Student *chf*'s have an elementary form for odd integer

Table 1. Examples of odd integer order ($\nu = 2n + 1$), dimensionless and reduced Student laws.

ν	n	$f(x\|2n+1)$	$\varphi(u\|2n+1)$
1	0	$\frac{1}{\pi}(1+x^2)^{-1}$	$\mathrm{e}^{-\|u\|}$
3	1	$\frac{2}{\pi}(1+x^2)^{-2}$	$\mathrm{e}^{-\|u\|}(1+\|u\|)$
5	2	$\frac{8}{3\pi}(1+x^2)^{-3}$	$\mathrm{e}^{-\|u\|}(1+\|u\|+\frac{1}{3}\|u\|^2)$
7	3	$\frac{16}{5\pi}(1+x^2)^{-4}$	$\mathrm{e}^{-\|u\|}(1+\|u\|+\frac{2}{5}\|u\|^2+\frac{1}{15}\|u\|^3)$

values of the parameter ν. In fact from equation (17) we have for $\nu = 2n+1$ with $n = 0, 1, \ldots$ and with $\ell = n - j$

$$f(x|2n+1) = \frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{1}{1+x^2}\right)^{n+1} = \frac{(2n)!!}{\pi(2n-1)!!}\left(\frac{1}{1+x^2}\right)^{n+1}$$

$$\varphi(u|2n+1) = 2\frac{|u|^{n+\frac{1}{2}}K_{n+\frac{1}{2}}(|u|)}{2^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)} = \mathrm{e}^{-|u|}\sum_{\ell=0}^{n}\frac{n!}{(2n)!}\frac{(2n-\ell)!}{(n-\ell)!}\frac{(2|u|)^{\ell}}{\ell!}$$

so that the *chf* is just an exponential times a polynomial in $|u|$ (see table 1 for a few explicit examples). The first case $n = 0$, $\nu = 1$ is just the stable, reduced Cauchy law $\mathcal{C}(1)$ which produces the well-known Cauchy process. We can then look at the explicit time evolution of the first nonstable case by taking the $n = 1$, $\nu = 3$ law, namely the $\mathcal{T}(3)$-process with *pdf*

$$\begin{aligned} p(x,t|3) &= \frac{1}{\pi}\int_0^{+\infty}\cos(ux)\,\mathrm{e}^{-tu}(1+u)^t\,\mathrm{d}u \\ &= \mathrm{Re}\left\{\frac{1}{\pi}\int_0^{+\infty}\mathrm{e}^{-(t+\mathrm{i}x)u}(1+u)^t\,\mathrm{d}u\right\}. \end{aligned}$$

By taking then

$$Q(a,z) = \frac{1}{\pi}\int_0^{+\infty}\mathrm{e}^{-zu}(1+u)^{a-1}\,\mathrm{d}u = \frac{1}{\pi}\frac{e^z}{z^a}\Gamma(a,z)$$

$$\Gamma(a,z) = \int_z^{+\infty}\mathrm{e}^{-w}w^{a-1}\,\mathrm{d}w, \qquad \Gamma(a,0) = \Gamma(a),$$

where $\Gamma(a,z)$ is the incomplete Gamma function (Abramowitz and Stegun 1968), we can also write

$$p(x,t|3) = \mathrm{Re}\{Q(t+1,t+\mathrm{i}x)\} = \mathrm{Re}\left\{\frac{\mathrm{e}^{t+\mathrm{i}x}\Gamma(t+1,t+\mathrm{i}x)}{\pi(t+\mathrm{i}x)^{t+1}}\right\}. \tag{29}$$

This new closed form (29) of the increment laws of the Student process with $\nu = 3$ is now explicitly given for every time $t > 0$: in the following sections we will try to analyse its properties.

5.2. *Asymptotic behaviour of the $\mathcal{T}(3)$-process*

Since the Student laws are not closed under convolution we know that $p(x,t|3)$ coincides with a Student law only for $t = 1$. A first question is then to check if, that notwithstanding, some important property of the $t = 1$ distribution is preserved along the evolution. In fact we will see in the following that for an arbitrary fixed, finite $t > 0$ the asymptotic behaviour of $p(x,t|3)$ for large x is always infinitesimal at the same order $|x|^{-4}$ of the original $\mathcal{T}(3)$.

Proposition 5.1. *If* $p(x,t|3)$ *is the pdf (29) of a Student* $\mathcal{T}(3)$*-process, then*

$$p(x,t|3) = \frac{2t}{\pi x^4} + o(|x|^{-4}), \qquad |x| \to +\infty$$

for every given $t > 0$.

Proof. Let us remember first of all that by repeated integration by parts of the incomplete Gamma function we get the following recurrence formula: for a given $a > 0$ and $n = 1, 2, \ldots$

$$Q(a,z) = \frac{1}{\pi}\frac{e^z}{z^a}\Gamma(a,z) = \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{\Gamma(a)}{\Gamma(a-k)}\frac{1}{z^{k+1}} + R_n(a,z)$$

$$R_n(a,z) = \frac{1}{\pi}\frac{e^z}{z^a}\frac{\Gamma(a)}{\Gamma(a-n)}\Gamma(a-n,z)$$

where, from a classical result about this asymptotical expansion (Gradshteyn and Ryzhik 1980), the remainder $R_n(a,z)$ is an infinitesimal of order greater than n

$$|R_n(a,z)| = O(|z|^{-n-1}), \qquad |z| \to +\infty.$$

Then, for $a = t+1$ and $z = t + \mathrm{i}x$ with an arbitrary but fixed $t > 0$, we will have in the limit $|x| \to +\infty$

$$|\mathrm{Re}\{R_n(t+1, t+\mathrm{i}x)\}| \leqslant |R_n(t+1, t+\mathrm{i}x)| = O(|x|^{-n-1}).$$

Now take $n = 4$: from the previous expansion and equation (29) we have for $|x| \to +\infty$

$$\begin{aligned} p(x,t|3) &= \mathrm{Re}\{Q(t+1, t+\mathrm{i}x)\} \\ &= \frac{1}{\pi}\sum_{k=0}^{3}\frac{\Gamma(t+1)}{\Gamma(t-k+1)}\frac{\mathrm{Re}\{(t-\mathrm{i}x)^{k+1}\}}{(t^2+x^2)^{k+1}} + o(|x|^{-4}) \end{aligned}$$

while from a direct calculation of the real parts we will find that the higher powers exactly cancel away from the numerator so that the leading asymptotic term for $|x| \to +\infty$ is of the order $|x|^{-4}$; more precisely we have

$$\begin{aligned} \sum_{k=0}^{3}\frac{\Gamma(t+1)}{\Gamma(t-k+1)}\frac{\mathrm{Re}\{(t-\mathrm{i}x)^{k+1}\}}{(t^2+x^2)^{k+1}} &= \frac{2tx^4 - 4t^3(t^2-5t+3)x^2 + 2t^5(2t^2-2t+1)}{(t^2+x^2)^4} \\ &= \frac{2t}{x^4} + o(|x|^{-4}) \end{aligned}$$

giving finally the statement in our proposition. □

It must be remarked that the previous result is true for an arbitrary finite, fixed time t. For diverging t, however, the reduced law of the process approaches a Gaussian: let $X(t)$ be our $\mathcal{T}(3)$-process with *pdf* $p(x,t|3)$; then we know that

$$\mathbf{E}[X(t)] = 0, \qquad \mathbf{Var}[X(t)] = \mathbf{E}[X^2(t)] = t$$

so that $t^{-1/2}X(t)$ is a centred, reduced rv for every t. A simple look at the *chf*'s will then shows that in distribution we have

$$\frac{X(t)}{\sqrt{t}} \xrightarrow{d} \mathcal{N}(0,1), \qquad t \to +\infty$$

since for large values of t and arbitrary fixed u

$$[\varphi(u/\sqrt{t}|3)]^t = \left[\mathrm{e}^{-|u|/\sqrt{t}}\left(1 + \frac{|u|}{\sqrt{t}}\right)\right]^t \longrightarrow \mathrm{e}^{-u^2/2}, \qquad t \to +\infty.$$

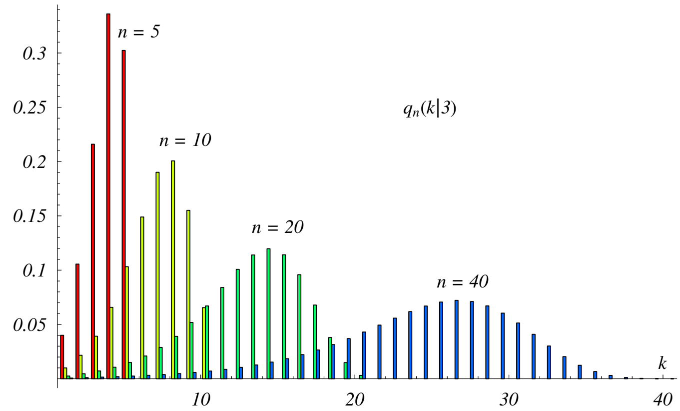

Figure 1. Mixture weights of the integer time ($t = n$) components for a Student process with $\nu = 3$.

5.3. The $\mathcal{T}(3)$-process distribution at integer times $t = n$

To understand the time evolution of $p(x, t|3)$ we can analyse the former of this *pdf* for integral values of the time $t = n = 1, 2, \ldots$ since in this case the distributions have explicit elementary expressions. Of course $p(x, n|3)$ is nothing else than the distribution of the sum of n independent $\mathcal{T}(3)$ rv's, so that the following proposition can also be seen as a new result about the nth convolution of the law $\mathcal{T}(3)$.

Proposition 5.2. *For $n = 1, 2, \ldots$ we have (within the notations of the present section)*

$$p(x, n|3) = \sum_{k=0}^{n} f(x|2k+1, n)q_n(k|3)$$

$$q_n(k|3) = \frac{(-1)^k}{2k+1}\sum_{j=0}^{2k+1}\binom{n}{j}\binom{2k+1}{j}\binom{j}{k}(j+1)!\left(\frac{-1}{2n}\right)^j$$

where $q_n(k|3)$ is a discrete probability distribution taking (strictly) positive values only for $k = 1, 2, \ldots, n$ (in particular $q_n(0|3) = 0$ for every n) and such that

$$\sum_{k=1}^{n}\frac{q_n(k|3)}{2k-1} = \frac{1}{n}$$

Proof 2. see appendix A □

The meaning of proposition 5.2 is then that (at least) at integral times $t = n = 1, 2, \ldots$ the marginal one-dimensional *pdf* $p(x, n|3)$ of the $\mathcal{T}(3)$-Student process is a mixture (convex combination) of Student *pdf*'s (26) $f(x|\nu, \delta)$ with

- odd integer orders $\nu = 2k + 1$ with $k = 0, 1, \ldots,$
- integer scaling factors $\delta = n$,
- relative weights $q_n(k|3)$ such that $q_n(0|3) = 0$, so that no Student distribution of order smaller than $\nu = 3$ appears in the mixture.

In other words they are mixtures of $\mathcal{T}(2k + 1, n)$ laws. The distributions $q_n(k|3)$ are a new kind of discrete probability laws whose bar diagrams at different times $t = n$ are displayed in figure 1. They show how the weight of the higher order Student distributions grows with the time, but also that, this notwithstanding, the lowest order ($\nu = 3$) distribution is always present—albeit with dwindling importance—with a nonzero weight. We see at once that this new result is coherent with proposition 5.1 and explicitly shows how the asymptotic behaviour is kept $|x|^{-4}$ all along the time evolution: in fact in the mixture representing $p(x, n|3)$ the lowest order Student distribution always is—albeit with dwindling weight—that with $\nu = 3$ which asymptotically behaves as $|x|^{-4}$; all the other components in the mixture are instead faster infinitesimals. The importance of the higher orders, however, grows with the time. This is exactly the behaviour recently observed in complex dynamical systems used to simulate the behaviour of intense beams of charged particles in accelerators (Vivoli *et al* 2006). Due to their mutual interactions these particles follow irregular paths, and a statistical analysis shows that the distribution of the increments follows an almost Gaussian distribution in its central part, and a Student $\mathcal{T}(3, \delta)$ distribution on the tails with a $|x|^{-4}$ decay rate. This suggests that the beam particles follow a $\mathcal{T}(3, \delta)$ Lévy process which is observed at a time scale (Δt) large when compared to some characteristic time T of the process, but finite and fixed so that the increment distribution shows two different regimes (Gaussian and $|x|^{-4}$) in the two regions.

The results presented in propositions 5.1 and 5.2 that the *pdf* of a Lévy–Student process is a suitable finite mixture of other Student *pdf*'s of different types has been proved here only in the particular conditions chosen for our demonstration. It suggests however a possible generalization: it is fair in fact to put forward the conjecture that every Lévy–Student process at every time will have a marginal one-dimensional *pdf* which is a mixture of other Student *pdf*'s, but not necessarily (as in our particular case) of a finite number of odd integer indices Student *pdf*'s. In other words, by keeping always the same notation, the *pdf* $p(x, t|\nu_0)$ could be a (possibly continuous) mixture of Student *pdf*'s $f(x|\nu, \delta)$ through a (possibly continuous) distribution $q_t(\nu|\nu_0)$. Finally, in order to preserve the result of proposition 5.1, we could also conjecture that $q_t(\nu|\nu_0)$ gives probability zero in the mixture to every Student law with $\nu < \nu_0$. If some form of this conjecture shows up to be true this would determine some new family of randomized Student distributions which is closed under convolution.

5.4. The Lévy triplet for a $\mathcal{T}(3)$-process

We will finally calculate the elements of the Lévy triplet for a $\mathcal{T}(3)$-process from the formulae (7), (8) and (9). First of all, due to the $\mathcal{T}(3)$ law symmetry, we already know that $A = 0$; then we must recall that the *chf* of the $\mathcal{T}(3)$ law is

$$\varphi(u|3) = \mathrm{e}^{-|u|}(1 + |u|) \tag{30}$$

so that by a direct calculation we get an explicit expression of the Lévy triplet with $T = 1$ (for details on the derivation see appendix B)

$$A = 0, \qquad B = 0, \qquad W(z) = \frac{1 - |z|\,(\sin|z|\mathrm{ci}|z| - \cos|z|\mathrm{si}|z|)}{\pi z^2}, \tag{31}$$

where the sine and the cosine integral functions are (Gradshteyn and Ryzhik 1980)

$$\mathrm{si}\,x = -\int_x^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t, \qquad \mathrm{ci}\,x = -\int_x^{+\infty} \frac{\cos t}{t}\,\mathrm{d}t.$$

A plot of $W(z)$ is shown in figure 2 where it is also compared with the analogous density (24) for a VG process. The behaviour of $W(z)$ at the origin and at the infinity is

$$W(z) = \begin{cases} z^{-2} + o(z^{-2}), & z \to 0^+; \\ 2z^{-4} + o(z^{-4}), & z \to +\infty. \end{cases}$$

In particular remark that near the origin it has the same behaviour of the Lévy density for the Cauchy process in (11), while it asymptotically behaves exactly as the $\mathcal{T}(3)$ distribution.

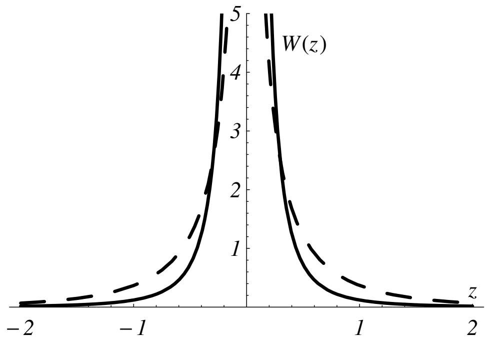

Figure 2. Plot of the (reduced and dimensionless) Lévy densities for a Student with $\nu = 3$ (solid line) and for a VG (dashed line) process.

We could then also conjecture here that the $W(z)$ function of a generic $\mathcal{T}(\nu, \delta)$-process will always have a z^{-2} behaviour for $z \to 0^+$, and a $|z|^{-\nu-1}$ behaviour for $|z| \to +\infty$. From (31) we also see that, always with $T = 1$, the PIDE (6) for a $\mathcal{T}(3)$-process takes the particular form

$$\partial_t p(x, t) = \lim_{\epsilon \to 0^+} \int_{|z| \geqslant \epsilon} W(z)[p(x + z, t) - p(x, t)]\, \mathrm{d}z$$

with $W(z)$ given in (31). Finally, inspection into the Lévy–Khinchin formula (25) for the *chf* (30) immediately gives as a byproduct a previously unknown way to calculate a nontrivial integral:

$$\frac{2}{\pi} \int_0^{+\infty} \frac{\sin z \,\mathrm{ci}\, z - \cos z \,\mathrm{si}\, z}{z} (1 - \cos uz)\, \mathrm{d}z = \log(1 + |u|). \tag{32}$$

6. Pathwise properties and simulations

Both the classes of processes analysed in this paper do not have a Brownian component ($B = 0$) in their Lévy decomposition which is in fact reduced to its jumping part and has the form (Cont and Tankov 2004, Øksendal and Sulem 2005)

$$X(t) = \int_{|z| \geqslant 1} z N(t, \mathrm{d}z) + \lim_{\epsilon \to 0^+} \int_{\epsilon \leqslant |z| < 1} z \tilde{N}(t, \mathrm{d}z) \qquad \tilde{N}(t, U) = N(t, U) - \frac{t}{T} \nu(U),$$

where U is a Borel set $U \subset \mathbb{R}$, $N(t, U)$ is the jump measure of the process, namely is the number of the (nonzero) jumps of size in U occurring in $[0, t]$, and $\nu(U) = \mathbf{E}[N(1, U)]$ is the Lévy measure of the process. In fact $N(t, U)$ is a Poisson process of intensity $\nu(U)$ and $\tilde{N}(t, U)$ is the corresponding compensated Poisson process. The function $W(x)$ introduced in the previous sections of this paper plays the role of a density for the Lévy measure in the sense that $\nu(\mathrm{d}x) = T W(x)\, \mathrm{d}x$, so that we have all the elements to characterize the Lévy decompositions of our processes. In particular, due to the nature of the singularities of the $W(x)$ functions in $x = 0$, it is possible to see that both the VG and the $\mathcal{T}(3, \delta)$ processes (as well as the Cauchy process) have infinite activity, namely that $\nu(\mathbb{R}) = +\infty$. In that event we know (Cont and Tankov 2004) that the set of jump times of every trajectory is countably infinite and dense in $[0, +\infty]$. This property, together with the continuous distributions of the jump sizes, accounts for the fact that at first sight the (simulated) samples of both a VG and a $\mathcal{T}(3, \delta)$ process do not look very different from that of a Wiener process, in particular when we compare just the free trajectories of these processes. Then to better see the respective pathwise

Table 2. The unit variance laws and *pdf*'s of the increments used in producing the samples of figure 3.

(a)	(b)	(c)
$\mathcal{N}(0,1)$	$\mathcal{VG}(1,\sqrt{2})$	$\mathcal{T}(3,1)$
$\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$	$\frac{1}{\sqrt{2}}\,\mathrm{e}^{-\sqrt{2}\lvert x\rvert}$	$\frac{2}{\pi}\,\frac{1}{(1+x^2)^2}$

characteristics it will be useful to introduce some Lévy diffusions, namely the solutions of other SDE driven by a Lévy process $X(t)$ (Protter 2004, Applebaum 2004, Øksendal and Sulem 2005). If $X(t)$ is a pure jump Lévy process, let us consider the Lévy diffusions $Y(t)$ solution of the SDE

$$\mathrm{d}Y(t) = \alpha(t, Y(t))\,\mathrm{d}t + \mathrm{d}X(t)$$
$$\mathrm{d}X(t) = \int_{|z|\geqslant 1} zN(\mathrm{d}t,\mathrm{d}z) + \lim_{\epsilon\to 0^+}\int_{\epsilon\leqslant|z|<1} z\tilde{N}(\mathrm{d}t,\mathrm{d}z)$$

which is nothing else than a deterministic dynamic system $\dot{y}(t) = \alpha(t, y(t))$ perturbed by a jump noise $X(t)$. The simplest case is that of a linear force $\alpha(y) = -ky$ giving rise to non-Gaussian Ornstein-Uhlenbeck (OU) processes (see for example Barndorff–Nielsen and Shephard (2001), Cont and Tankov (2004))

$$\mathrm{d}Y(t) = -kY(t)\,\mathrm{d}t + \mathrm{d}X(t). \tag{33}$$

The usual, Gaussian OU process, on the other hand, is the solution of a SDE where the noise $B(t)$ is completely Brownian with no jump component:

$$\mathrm{d}Y(t) = -kY(t)\,\mathrm{d}t + \mathrm{d}B(t). \tag{34}$$

We can compare now the samples of OU-type processes driven either by a Brownian noise, or by a pure jump noise as the VG and the $\mathcal{T}(3,\delta)$ processes. To do that we will produce samples of 5000 steps by using reduced and dimensionless versions of our distributions that we will take of unit variance. In particular we will suppose that for time intervals $\Delta t = T$ the laws of the noise increments are that reproduced in table 2. Of course the choice of $\Delta t = T$ is instrumental because the VG and the Student laws have distributions of elementary form only for $\Delta t = nT$ with n integer (and particularly simple for $n = 1$), as we have seen in the previous sections. It is not so easy, on the other hand, to produce our pure jump driven trajectories at other time scales, in particular for time scales which are fractions of T. At first sight we could think to overcome this difficulty by arbitrarily changing the value of T, but we should remember from our previous discussion (section 2) that our pure jump processes are not scale invariants, so that different values of T produce different processes. Examples of simulated samples of these processes are produced by discretizing our SDE and are shown in figure 3 as functions of the dimensionless time $\tau = t/T$. The parts (a), (b) and (c) show trajectories produced by our three different SDE's: while (a) is a typical sample of an OU process solution of the SDE (34) driven by a normal Brownian motion, the parts (b) and (c) display typical trajectories produced by the SDE (33) driven by respectively a VG noise and a Student noise. The plots are on the same spatial scale and we can see the jumping nature of the non-Gaussian noises from the fact that, while trajectory (a) is rather strictly confined inside the region determined by the restoring force $-ky$, the trajectory (b), and above all the trajectory (c) show clearly random spikes going outside the confining region. These spikes are produced by the jumps of the driving processes, and the fact that the Student (c) spikes are larger than that of the VG (b) case depends on the fact that the VG distribution has exponential

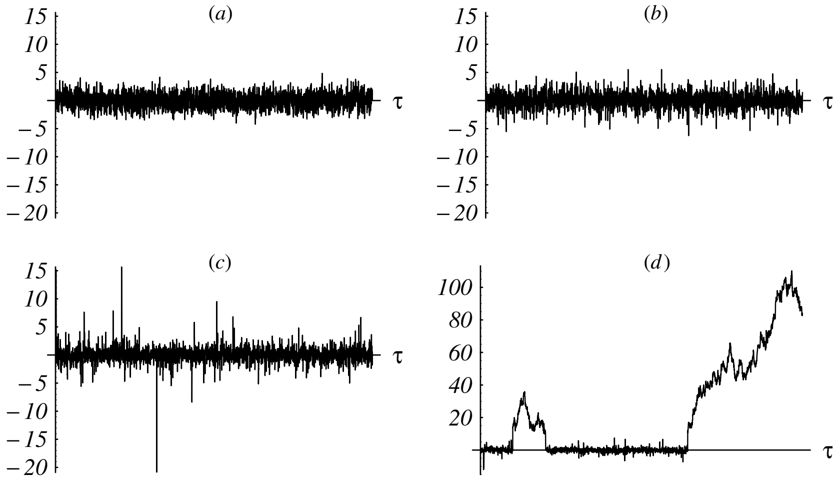

Figure 3. Samples of OU-type diffusions ($\tau = t/T$): (a) usual OU process driven by Gaussian Brownian motion; (b) OU-type process driven by a VG Lévy noise; (c) OU-type process driven by a Student Lévy noise; (d) OU-type process with Student noise and restoring force of limited range.

tails which—albeit longer than the Gaussian tails—are much shorter than the power tails of a Student distribution (see also the corresponding asymptotic behaviour of the Lévy densities $W(z)$ displayed in sections 4 and 5.4). The size of the spikes can also be put in evidence by cutting the restoring force of the SDE's to a finite length, namely by considering the solutions of

$$\mathrm{d}Y(t) = \alpha(Y(t))\,\mathrm{d}t + \mathrm{d}X(t)$$

$$\alpha(y) = \begin{cases} -ky, & \text{for} \quad |y| \leqslant q; \\ 0, & \text{for} \quad |y| > q; \end{cases} \qquad q > 0.$$

In this case the restoring force acts only when the process lies in $[-q, q]$, while the process is completely free outside this region. Hence when the process jumps beyond the boundaries in $y = \pm q$ it begins to diffuse freely drifting away from the bounding region. Occasionally, however, it can also be recaptured by the binding force. All these features are represented in part (d) of figure 3 which displays the trajectory of a Student driven OU-type process with a limited range of the force. To compare it with the other two cases we must now look at the different values of q that make an escape reasonably likely: while to let an OU Gaussian process to escape is necessary to have a rather small value of q, evasions are likely in the VG case for larger, and in the Student case even for much larger, q values.

7. Conclusions

We have studied in this paper a few examples of nonstable, infinitely divisible processes, and in particular we have explicitly written down their evolution equations and the laws of the increments which are the germ of the corresponding markovian evolutions. In particular we focused our attention on the Student processes and we presented a new explicit form of their transition functions. Since the Student family of laws is infinitely divisible, but nonclosed under convolution the distribution of the corresponding Lévy–Student process is a Student distribution only at the one particular time. Along the evolution, instead, the process distribution is no longer a simple Student distribution. We have shown in the previous sections that, this notwithstanding, at least in the case of a specific type of Student distribution (with finite variance), and at least in an infinite sequence of equidistant time instants the process transition law is a mixture of a finite number of Student laws given by means of a new kind of discrete probability distribution. This prompts the conjecture that in fact while the Student family is not closed under convolution, some family of mixtures of Student distributions can possibly be closed. On the other hand, while it is easy to show that for large values of time the reduced increment law tends to be normal (as it should be since we are dealing with finite variance distributions), we have also emphasized that for a finite (albeit large) time the asymptotic behaviour always is the same as that of the Student distribution at the unit time. This behaviour has been put in evidence by Vivoli *et al* (2006) in their model for halo in particle beams, and we have put forward the conjecture that this could also be a more general behaviour of the Student processes. This last remark is interesting also in connection with a possible generalization of the stochastic mechanics that we mentioned in the section 1 and that will be the argument of forthcoming research.

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Appendix A. Proof of proposition 5.2

For $a = n + 1$ and $n = 1, 2, \ldots$ the incomplete Gamma functions have a finite elementary expression (Gradshteyn and Ryzhik 1980) so that

$$Q(n+1, z) = \frac{e^z}{z^{n+1}}\Gamma(n+1, z) = \sum_{j=0}^{n} \frac{n!}{(n-j)!}\frac{1}{z^{j+1}}$$

and hence we get

$$\begin{aligned} p(x, n|3) &= \frac{1}{\pi}\mathrm{Re}\{Q(n+1, n+\mathrm{i}x)\} = \frac{1}{\pi}\sum_{j=0}^{n}\frac{n!}{(n-j)!}\mathrm{Re}\left\{\frac{1}{(n+\mathrm{i}x)^{j+1}}\right\} \\ &= \frac{1}{\pi}\sum_{j=0}^{n}\frac{n!}{(n-j)!}\frac{1}{(n^2+x^2)^{j+1}}\mathrm{Re}\left\{\sum_{m=0}^{j+1}\binom{j+1}{m}(-\mathrm{i}x)^m n^{j-m+1}\right\} \\ &= \frac{1}{\pi}\sum_{j=0}^{n}\frac{n!}{(n-j)!}\frac{1}{(n^2+x^2)^{j+1}}\sum_{2\ell=0}^{j+1}\binom{j+1}{2\ell}(-1)^\ell x^{2\ell} n^{j-2\ell+1}, \end{aligned}$$

where it is understood that the second sum is extended to all the integer values of ℓ such that $0 \leqslant 2\ell \leqslant j+1$, namely: if j is even then $\ell = 0, 1, \ldots, \frac{j}{2}$; if j is odd then $\ell = 0, 1, \ldots, \frac{j+1}{2}$. A little manipulation and the use of equation (26) then give

$$\begin{aligned} p(x, n|3) &= \frac{1}{\pi}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^{j+1}}\left(\frac{n^2}{n^2+x^2}\right)^{j+1}\sum_{2\ell=0}^{j+1}\binom{j+1}{2\ell}(-1)^\ell\left(\frac{x^2}{n^2}\right)^\ell \\ &= \frac{1}{\pi}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^j}\sum_{2\ell=0}^{j+1}\binom{j+1}{2\ell}\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{(-1)^m}{n}\left(\frac{n^2}{n^2+x^2}\right)^{j-m+1} \end{aligned}$$

$$= \frac{1}{\pi}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^j}\sum_{2\ell=0}^{j+1}\binom{j+1}{2\ell}\sum_{m=0}^{\ell}(-1)^m\binom{\ell}{m}$$
$$\times B\left(\frac{1}{2}, j-m+\frac{1}{2}\right) f(x|2(j-m)+1,n)$$

with $f(x|\nu)$ defined in (26). Now by exchanging the order of the last two sums (with the previous conventions about the range of the indexes ℓ and m) we have with $k = j - m$

$$p(x,n|3) = \frac{1}{\pi}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^j}\sum_{2m=0}^{j+1}(-1)^m f(x|2(j-m)+1,n)$$
$$\times B\left(\frac{1}{2}, j-m+\frac{1}{2}\right)\sum_{2\ell=2m}^{j+1}\binom{j+1}{2\ell}\binom{\ell}{m}$$
$$= \frac{1}{\pi}\sum_{j=0}^{n}\binom{n}{j}\frac{j!}{n^j}\sum_{2m=0}^{j+1}(-1)^m f(x|2(j-m)+1,n)$$
$$\times B\left(\frac{1}{2}, j-m+\frac{1}{2}\right)\frac{2^{j-2m}(j+1)(j-m)!}{m!(j-2m+1)!}$$
$$= \sum_{j=0}^{n}\binom{n}{j}\frac{1}{(2n)^j}\sum_{2m=0}^{j+1}\frac{(-1)^m(j+1)!(2j-2m)!}{(j-m)!m!(j-2m+1)!}f(x|2(j-m)+1,n)$$
$$= \sum_{j=0}^{n}\binom{n}{j}\frac{1}{(2n)^j}\sum_{2k\geqslant j-1}^{2j}\frac{(-1)^{j-k}(j+1)!(2k)!}{k!(j-k)!(2k-j+1)!}f(x|2k+1,n),$$

where it is understood that the second sum extends over all the k values such that $j-1 \leqslant 2k \leqslant 2j$, namely: for odd j we have $k = \frac{j-1}{2}, \ldots, j$, while for even j we have $k = \frac{j}{2}, \ldots, j$. Finally, by exchanging again the sums and by adopting the convention that a binomial symbol $\binom{a}{b}$ always is zero whenever the limitation $b \leqslant a$ is not verified, we have the results of proposition 5.2

$$p(x,n|3) = \sum_{k=0}^{n} f(x|2k+1,n)q_n(k|3)$$
$$q_n(k|3) = \frac{(-1)^k}{2k+1}\sum_{j=0}^{2k+1}\binom{n}{j}\binom{2k+1}{j}\binom{j}{k}(j+1)!\left(\frac{-1}{2n}\right)^j.$$

Since the distribution of our Student process is now represented as a linear combination of the Student $\mathcal{T}(2k+1,n)$ *pdf*'s, $p(x,n|3)$ turns out to be a (randomized, Feller 1971) mixture, and the coefficient $q_n(k|3)$ of this combination must satisfy

$$q_n(k|3) \geqslant 0, \qquad \sum_{k=0}^{n} q_n(k|3) = 1$$

with $q_n(0|3) = 0$ for every n, as can be seen by direct calculation. Hence we have that $q_n(k|3)$ is a discrete probability distribution taking nonzero values only for $k = 1, 2, \ldots, n$. Finally by remembering that our Student process has zero expectation and variance $t = n$, and taking also into account the equation (21), we can write

$$n = \int_{-\infty}^{+\infty} x^2 p(x, n|3)\,\mathrm{d}x = \sum_{k=0}^{n} q_n(k|3) \int_{-\infty}^{+\infty} x^2 f(x|2k+1, n)\,\mathrm{d}x$$
$$= \sum_{k=0}^{n} q_n(k|3) \frac{n^2}{2k-1}$$

so that we immediately get also the last result in our proposition.

Appendix B. Derivation of equation (31)

From (9) and (30) we have for a $\mathcal{T}(3)$-process that

$$B = \frac{1}{\pi} \lim_{\epsilon\to 0^+} \lim_{M\to+\infty} \int_{-M}^{M} \frac{-u}{1+|u|} \frac{u\epsilon\cos u\epsilon - \sin u\epsilon}{u^2}\,\mathrm{d}u$$
$$= \frac{2}{\pi} \lim_{\epsilon\to 0^+} \int_{0}^{+\infty} \frac{\sin u\epsilon - u\epsilon\cos u\epsilon}{u(1+u)}\,\mathrm{d}u$$
$$= \frac{2}{\pi} \lim_{\epsilon\to 0^+} \left[\frac{\pi}{2} - (\mathrm{ci}\,\epsilon - \epsilon\,\mathrm{si}\,\epsilon)\sin\epsilon + (\epsilon\,\mathrm{ci}\,\epsilon + \mathrm{si}\,\epsilon)\cos\epsilon\right] = 0,$$

where the sine and the cosine integral functions are defined in the text: hence, as for the Cauchy and the VG processes, the Brownian part is absent also in this Student process. As for the Lévy density $W(z)$, from (7) we get

$$W(z) = \frac{1}{2\pi\mathrm{i}z} \lim_{M\to+\infty} \int_{-M}^{M} \frac{-u}{1+|u|}\,\mathrm{e}^{-\mathrm{i}uz}\,\mathrm{d}u$$
$$= \frac{1}{\pi|z|} \lim_{M\to+\infty} \int_{0}^{M} \frac{u}{1+u} \sin(u|z|)\,\mathrm{d}u$$
$$= \frac{1 + |z|(\cos|z|\mathrm{si}|z| - \sin|z|\mathrm{ci}|z|)}{\pi z^2}$$

so that for our $\mathcal{T}(3)$-process we finally have (31).

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