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DYNAMICAL CONTROL OF THE HALO IN PARTICLE BEAMS: A STOCHASTIC–HYDRODYNAMIC APPROACH

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In this paper we describe the beam distribution in particle accelerators in the framework of a stochastic–hydrodynamic scheme. In this scheme the possible reproduction of the halo after its elimination is a consequence of the stationarity of the transverse distribution which plays the role of an attractor for every other distribution. The relaxation time toward the halo is estimated, and a few examples of controlled transitions toward a permanent halo elimination are discussed.

1. Introduction

In high intensity beams of charged particles, proposed in recent years for a wide variety of accelerator-related applications, it is very important to keep at low level the beam loss to the wall of the beam pipe, since even small fractional losses in a high-current machine can cause exceedingly high levels of radioactivation. It is now widely believed that one of the relevant mechanisms for these losses is the formation of a low intensity beam halo more or less far from the core. These halos have been observed[1] or studied in experiments,[2] and have also been subjected to an extensive simulation analysis.[3] For the next generation of high intensity machines it is however still necessary to obtain a more quantitative understanding not only of the physics of the halo, but also of the beam transverse distribution in general.[4–6] In fact "because there is not a consensus about its definition, halo remains an imprecise term"[7] so that several proposals have been put forward for its description.

The charged particle beams are usually described in terms of classical dynamical systems. The standard model is that of a collisionless plasma where the corresponding dynamics is embodied in a suitable phase space (see for example Ref. 8). We propose and develop a different approach.[9,10] a model for the halo formation in particle beams based on the idea that the trajectories are samples of a conservative stochastic process,[11] rather than usual deterministic (differentiable) trajectories.

As a first step to approach the halo problem, the method has been recently implemented to quantitatively investigate the nature, the size and the dynamical characteristics of a possible stationary beam halo.[12] In this paper we discuss the problem of the halo dynamics. At first, we give an estimate of the time needed by a non stationary, halo-free distribution to relax toward the stationary distribution with a halo when the dynamics is supposed to be frozen in the configuration that produces this halo. Since this relaxation time depends on the parameters of our beam, a comparison with possibly measured phenomenological time could constitute a good check on the soundness of the model. Then we begin to analyze possible transitions from a beam with halo toward a halo–free one, and we put an emphasis on the possible dynamics which allows this halo elimination. Finally, we spend a few words about open problems.

2. Stochastic Beam Dynamics, Controlling Potentials and Stationary Halo Distributions

Time-reversal invariant diffusion processes are obtained by promoting deterministic kinematics to stochastic kinematics, and by adding a further dynamical prescription. The simpler, and most elegant, way to obtain the equations of such processes is to impose stochastic variational principles which generalise the usual ones of the deterministic mechanics to the case of diffusive kinematics.[13] This method can be applied to classical, conservative many-particle systems, whose complex dynamics can be effectively described by a representative particle performing stochastic trajectories. If ρ is the (normalized) density of the particles and $\mathbf{v}$ the current velocity, the stochastic variational principle leads to a gradient form for $\mathbf{v}$

$$m\mathbf{v}(\mathbf{r},\, t) = \nabla S(\mathbf{r},\, t)\,, \tag{1}$$

and to the couple of hydrodynamic equations

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v})\,, \tag{2}$$

$$\partial_t S + \frac{m}{2}\mathbf{v}^2 - 2mD^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + V(\mathbf{r}, t) = 0\,. \tag{3}$$

Here, D is the diffusion coefficient, and $V(\mathbf{r},\, t)$ is the external potential applied to the system. Due to the non-differentiability of the stochastic trajectories, it is not possible to define the standard velocity. One then introduce the forward velocity $\mathbf{v}_+$ (connected to the mean time-derivative from the right) and the backward velocity $\mathbf{v}_-$ (connected to the mean time-derivative from the left). In the conservative diffusions, the two velocities are exchanged under time-reversal. Furthermore, the current velocity is the balanced mean of $\mathbf{v}_{\pm}$; it describes the velocity of the centre of the density profile, and obviuosly reduces to the standard deterministic velocity if the noise is removed putting $D = 0$. In this last case, the two hydrodynamic equations reduces to the equations for an ideal fluid (continuity and Hamilton–Jacobi equations, respectively). Equation (2) can be also explicitely written in the form of Fokker–Planck equation

$$\partial_t \rho = -\nabla \cdot [\mathbf{v}_{(+)}\rho] + D\nabla^2 \rho \,, \tag{4}$$

formally associated to the Itô equation. It is finally important to remark that, introducing the representation[14]

$$\psi(\mathbf{r},\, t) = \sqrt{\rho(\mathbf{r},\, t)}\mathrm{e}^{iS(\mathbf{r},\, t)/\alpha} \,, \tag{5}$$

(with $\alpha = 2mD$) the coupled equations (2) and (3) are made equivalent to a single linear equation of the form of the Schrödinger equation in the function ψ, with the Planck action constant replaced by α:

$$i\alpha\partial_t \psi = -\frac{\alpha^2}{2m}\nabla^2\psi + V\psi \,. \tag{6}$$

We will refer to it as a Schrödinger-like (S-l) equation. In this formulation the phenomenological "wave function" ψ carries the information on the dynamics of both: the bunch density, and the velocity field of the bunch, since the velocity field is determined through equation (1) by the phase function $S(\mathbf{r},\, t)$. This shows that our procedure, starting from a different point of view, leads to a description formally analogous to that of the so called Quantum-like (Q–l) approaches to beam dynamics.[15] Our scheme allows also to implement a controlling procedure to drive on characteristic time scales an initial density profile to a desired, predetermined final density profile through an unitary evolution.[10] Let us suppose in fact that the probability density function (PDF) $\rho(\mathbf{r},\, t)$ is somehow given: think for example to the case of an engineered evolution from some initial PDF toward a final, required state with suitable characteristics. Then, the couple of hydrodynamic equations (2), (3) can be used reversing the point of view; we insert the known PDF and exploit the equations to compute the couple $(\mathbf{v}_+,\, V)$. As a consequence, $V(\mathbf{r},\, t)$ acquires the meaning of controlling potential which generates the desired solution in such a way that at any instant of time it satisfies the Schroedinger equation (6) associated to the potential itself (unitary evolution). We define now f as

$$\rho(\mathbf{r},\, t) = \mathrm{N} f(\mathbf{r},\, t) \,, \tag{7}$$

where N is a constant which leaves f dimensionless. We consider further that the relation between $\mathbf{v}$ and $\mathbf{v}_{\pm}$, leads to a gradient condition also for $\mathbf{v}_+$:

$$\mathbf{v}_{(+)}(\mathbf{r},\, t) = \nabla W(\mathbf{r},\, t) \,. \tag{8}$$

We see that relations (7), (8) and (1) give for the function S

$$S(\mathbf{r},\, t) = mW(\mathbf{r},\, t) - mD \ln f(\mathbf{r},\, t) - \theta(t) \,, \tag{9}$$

where θ is an arbitrary function of t only. Then, given f, it is straightforward to compute by the last three equations and by the hydrodynamic equations (2), (3) the general form of the controlling potential obtaining [10]

$$V(\mathbf{r},\,t) = mD^2\nabla^2 \ln f + mD(\partial_t \ln f + \mathbf{v}_{(+)} \cdot \nabla \ln f) - \frac{m}{2}\mathbf{v}^2_{(+)} - m\partial_t W + \dot{\theta}\,. \tag{10}$$

In the one-dimensional case we easily get

$$V(x,\,t) = mD^2\partial_x^2 \ln f + mD(\partial_t \ln f + v_{(+)}\partial_x \ln f) - \frac{m}{2}v^2_{(+)} - m\int_a^x \partial_t v_{(+)}(x',\,t)dx' + \dot{\theta}\,. \tag{11}$$

These expressions will be used in the next section to pick up a controlling potential which drives distribution with halo to a halo-free one.

3. Non-Stationary Distributions

In Ref. 12 we have considered our $\rho(\mathbf{r})$ as the stationary, ground state pdf of a suitable potential: the wave functions have no nodes and as a consequence we can also confidently state (see the general proof on the previous papers[16–18]) that, if we calculate $\mathbf{v}_{(+)}(\mathbf{r})$ and write down the corresponding FP equation, the distribution $\rho(\mathbf{r})$ will play the role of an attractor for every other distribution (non extremal with respect to a stochastic minimal action principle). If the accelerator beam is ruled by such an equation, this could imply that the halo cannot simply be wiped out by scraping away the particles that come out of the bunch core: in fact they simply will keep going out in the halo until the equilibrium is reached again since the distribution $\rho(\mathbf{r})$ is a stable attractor.

It is interesting to remark that this behavior could be not accounted for if the Q-l mechanisms would be considered a true quantum evolution. In fact a quantum evolution equation will always require that the distributions satisfy the FP evolution equation *and* a stochastic extremal principle. In this case, when we scrape away the halo, we have also supposed to change the dynamical situation so that $\mathbf{v}_{(+)}(\mathbf{r})$ no more corresponds to the previous effective potential and our initial $\rho(\mathbf{r})$ will no more behaves as an attractor. Here however we are simply in a mesoscopic context *formally* described by a Q-l mechanism; then the non extremal, relaxation processes can become observable. This is a consequence of the fact that now the physical quantity α playing the role of $\hbar$ is no more an universal constant.

We then are conjecturing here, at least approximately, that when we eliminate the halo we throw the system out of balance of the stochastic minimal action, and, since the halo represents just a tiny part of the beam:

(1) the velocity field changes slowly with respect to the characteristic relaxation times of the halo;

(2) the distributions will evolve (until the balance has been restored) only following a FP equation, and not a S-l equation.

As a consequence, in the phase of non extremal evolution the form of the effective potential will be irrelevant as long as the forward velocity field is given, since only this field enter the FP equation.

In this section we give an estimate of the time required for the relaxation of non extremal PDF's toward the equilibrium distribution. This is an interesting test for our conjecture since this relaxation time is fixed once the form of the forward velocity field is given; this is in turn fixed when the form of the halo distribution is given as in the previous paper,[12] and one could check if the estimate is in agreement with possible observed times. In order to do that we will study the FP equation corresponding to the given velocity field $\mathbf{v}_{(+)}$, but to speed the calculation we will limit ourselves to the one-dimensional case to simplify the calculation. Following the route of Ref. 12, we describe our initial stationary PDF by

$$\begin{gathered}\rho_0(x) = A\frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\left[1 + \frac{B}{\Gamma\left(q+\frac{1}{2}\right)}\left(\frac{x^2}{2\sigma^2}\right)^q\right]; \\ 0 \le A \le 0; B = \sqrt{\pi}\frac{1-A}{A} \ge 0, \quad q = 24\,,\end{gathered} \tag{12}$$

which is the one-dimensional version of the three-dimensional distribution with a halo introduced in Ref. 12. We are now interested in studying the non stationary solutions $\rho(x,\,t)$ of the FP equation

$$\begin{gathered}\partial_t\rho(x,\,t) = \frac{\alpha}{2m}\partial_x^2\rho(x,\,t) - \partial_x[v_{(+)}(x,\,t)\rho(x,\,t)]\,; \\ v_{(+)}(x) = \tfrac{\alpha}{2m}\frac{\rho_0'(x)}{\rho_0(x)} = \frac{\alpha}{m}\frac{u_0'(x)}{u_0(x)}, \quad u_0(x) = \sqrt{\rho_0(x)}\,,\end{gathered} \tag{13}$$

where the stationary velocity $v_{(+)}(x)$ is deduced from $\rho_0(x)$, and is chosen so that $\rho_0(x)$ is the stationary solution of the equation (13). This equation can now be put in self–adjoint form by means of the Ansatz $\rho(x,\,t) = \sqrt{\rho_0(x)}g(x,\,t) = u_0(x)g(x,\,t)$ so that Eq. (13) becomes

$$\begin{aligned}\partial_t g &= \frac{\alpha}{2m}\partial_x^2 g - \left(\frac{m}{2\alpha}v_{(+)}^2 + \frac{v_{(+)}'}{2}\right) \\ &= \frac{\alpha}{2m}\partial_x^2 g - \frac{1}{\alpha}\left(V - \frac{\alpha^2}{4m\sigma^2}\right)\,,\end{aligned} \tag{14}$$

where $V(x) = (\alpha^2/4m\sigma^2) + (\alpha^2/2m)(u_0''(x)/u_0(x))$ is the control potential. It is well–known[19] that this self–adjoint form allows now to expand the solutions of (13) in orthogonal eigenfunctions. In fact, if $g(x,\,t) = e^{-\Omega t}G(x)$, the equation (14) becomes

$$\Omega G(x) = \frac{\alpha}{2m}\left[\frac{u_0''(x)}{u_0(x)} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]G(x)\,, \tag{15}$$

or equivalently

$$EG(x) = \widehat{H}\, G(x) = \left[V(x) - \frac{\alpha^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right] G(x)\,, \quad E = \alpha\Omega + \frac{\alpha^2}{4m\sigma^2}\,, \tag{16}$$

which is formally a stationary, Schrödinger–like, eigenvalue equation for the potential $V(x)$. It is easy to check that $u_0(x)$ is eigenfunction of Eq. (15) with eigenvalue $\Omega_0 = 0$, and of Eq. (16) with eigenvalue $E_0 = \alpha^2/4m\sigma^2$. Now, since the general solution of Eq. (13) has the form $\rho(x,t) = u_0(x)\sum_{n=0}^{\infty} c_n G_n(x) e^{-\Omega_n t}$ where $G_n(x)$ and Ω_n are respectively eigenfunctions and eigenvalues, and since $\Omega_0 = 0$ while all the other eigenvalues are strictly positive (we suppose that they are ordered as an increasing sequence), all these solutions relax toward $\rho_0 = u_0^2$ with a relaxation time which is essentially given by $\tau_1 = {\Omega_1}^{-1}$. Thus, to evaluate the relaxation time, we are interested in an estimate of the order of magnitude of the eigenvalue Ω_1. To do that we first of all pass to a dimensionless formulation:

$$s = \frac{x}{\sigma\sqrt{2}}, \quad f(s) = f\left(\frac{x}{\sigma\sqrt{2}}\right) = \mathrm{N}G(x)\,, \tag{17}$$

$$w(s) = \sqrt{\frac{\sigma\sqrt{2\pi}}{A}}u_0(x) = e^{-s^2/2}\sqrt{1 + \frac{Bs^{2q}}{\Gamma\left(q+\frac{1}{2}\right)}}\,, \tag{18}$$

so that (16) becomes

$$\begin{aligned} \varepsilon f(s) &= \left[v(s) - \frac{\mathrm{d}^2}{\mathrm{d}s^2} \right] f(s)\,, \\ v(s) &= v\left(\frac{x}{\sigma\sqrt{2}}\right) = \frac{4m\sigma^2}{\alpha^2}V(x) = 1 + \frac{w''(x)}{w(x)}\,, \\ \varepsilon &= \frac{4m\sigma^2}{\alpha^2}\,E = \frac{4m\sigma^2}{\alpha^2}\Omega + 1\,, \end{aligned} \tag{19}$$

and its lowest eigenvalue is $\varepsilon_0 = 1$ with eigenfunction $w(s)$. Moreover a little algebra shows that the potential of our Schrödinger equation is

$$v(s) = s^2 + v_1(s)\,, \tag{20}$$

$$v_1(s) = \frac{qBs^{2(q-1)}}{Bs^{2q} + \Gamma\left(q+\frac{1}{2}\right)}\left[q - 1 - 2s^2 + \frac{q\Gamma\left(q+\frac{1}{2}\right)}{Bs^{2q} + \Gamma\left(q+\frac{1}{2}\right)} \right]\,, \tag{21}$$

so that is shows up to be a harmonic potential plus a correction which, as can be easily seen, remains bounded for all the s values. We can hence get our estimate of ε_1 just from the first perturbative correction ϵ_1 of the second eigenvalue μ_1 of the dimensionless equation for the harmonic oscillator

$$h''(s) + (\mu - s^2)h(s) = 0\,.$$

The second eigenvalue is 3 and, with the parameter values used to produce the Figs. 1, 2 and 3 ($A \approx 0.85$, $B \approx 0.32$ and $q = 24$), we get $\varepsilon_1 \approx 3.009$.

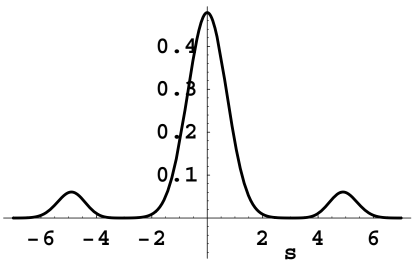

Fig. 1. Plot of the 1D density distribution (12) with a halo ring surrounding the beam core.

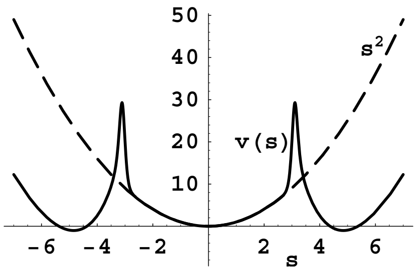

Fig. 2. The dimensionless potential for the 1D distribution of Fig. 1 (solid line), and for a harmonic oscillator (dashed line).

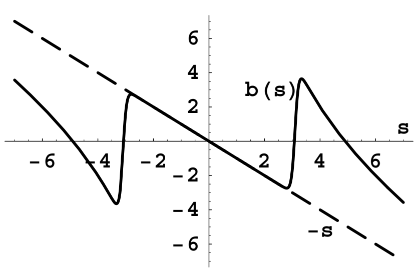

Fig. 3. The dimensionless velocity for the 1D distribution of Fig. 1 (solid line), and for a harmonic oscillator (dashed line).

This practically means that $\varepsilon_1 \approx \mu_1 = 3$ so that with the numerical value $(\alpha/4m\sigma^2 \approx 37.5 \text{ eV})$ estimated in Ref. 12 we have $\Omega_1 = (\varepsilon_1 - 1)(\alpha/4m\sigma^2) \approx \alpha/2m\sigma^2 = \omega$, and hence $\tau_1 \approx 2m\sigma^2/\alpha = \omega^{-1} \approx 10^{-8}$–$10^{-7}$ sec.

A different non-stationary problem consists in the analysis of some particular time evolution of the process with the aim of finding the dynamics that control it. For instance we would be interested in discussing possible evolutions starting from a PDF with halo toward a halo–free PDF, to find the dynamics that we are requested to apply in order to achieve this result. We have seen in Section 2 that, for a given evolution $\rho(\mathbf{r}, t)$, the corresponding control dynamics is given by the scalar potential (10) (or (11) in the one–dimensional case). Although very often its

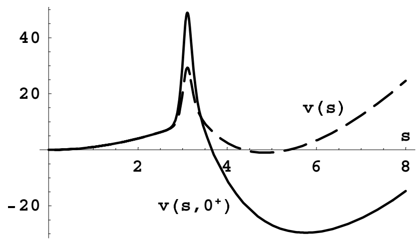

Fig. 4. Comparison between the potential $v(s)$ associated to the initial stationary distribution and the non stationary, controlling potential $v(s,0^+)$.

explicit expression cannot be given in terms of elementary functions, it must however be remarked that we can have still interesting, time dependent, non-stationary solutions of a FP equation (4) even when the velocity is stationary. For example it is well known that for the Ornstein–Uhlenbeck process the forward velocity field is a stationary, linear function of the position, and every initial PDF is then driven by the Chapman–Kolmogorov equation toward a stationary, asymptotic, Gaussian solution. In this case every possible evolution is by definition associated with a very simple velocity field, so that we can hope that the form of the potential can be elementary enough. We will now elaborate this example, and then consider a one–dimensional system, where x represents one of the transverse coordinates of the beam. Let us suppose that at the time $t = 0$ the PDF of our process is (12). If the process is supposed to be an Ornstein–Uhlenbeck one, the velocity field will simply be given by the stationary function $v_{(+)}^{ou}(x) = -(\alpha/2m\sigma^2)x$, and the evolution will be ruled by the Chapman–Kolmogorov equation

$$\rho(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,0)\rho(y)dy\,, \tag{22}$$

where the transition function is

$$\begin{gathered} p(x,t|y,0) = \frac{e^{-[x-\mu(t)]^2/2\nu^2(t)}}{\sqrt{2\pi\nu^2(t)}};\, \mu(t) = ye^{-\omega t}\,; \\ \nu^2(t) = \sigma^2(1-e^{-2\omega t});\quad \omega = \tfrac{\alpha}{2m\sigma^2}\,. \end{gathered} \tag{23}$$

A little algebra will show now that an application of Eq. (22) with Eq. (23) to the initial PDF (12) will give the following result:

$$\begin{aligned} \rho(x,t) &= A\frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} + (1-A)\frac{e^{-x^2/2\nu^2(t)}}{\nu(t)\sqrt{2\pi}}\lambda^{2q+1}(t) \\ &\quad \times \Phi\left(q+\frac{1}{2},\frac{1}{2};\frac{x^2\left[1-\lambda^2(t)\right]}{2\nu^2(t)}\right)\,, \\ \lambda^2(t) &= \frac{1-e^{-2\omega t}}{1+(p^2-1)e^{-2\omega t}}\,, \end{aligned} \tag{24}$$

where $\Phi(\alpha,\beta;z)$ is a confluent hypergeometric function. It is easy to see from the elementary properties of the hypergeometric functions that $\rho(x,+\infty) = \mathcal{N}(0,\sigma^2)$, namely the PDF asymptotically approaches a gaussian, halo-free distribution. On the other hand less immediate, but still easy enough, is to show that $\rho(x, 0^+) = \rho(x)$. A direct application of Eq. (11) allows now to calculate the control potential corresponding to Eq. (24): we prefer to plot it, because its expression, although available, is still so complicated that we do not consider useful to reproduce it here. As the PDF evolution (24) smoothly interpolates between the initial distribution with halo (12) and the final, asymptotic, gaussian, halo-free distribution, the corresponding control potential evolves from the three-hole form of Fig. 2 to that of a simple harmonic oscillator. However it must be remarked that the potential evolution is not completely smooth: the simulations show that $V(x, 0^+)$ is different from the stationary potential $V(x)$ obtained from the initial stationary condition. An example of the difference is shown in Fig. 4 where we have reduced $V(x,t)$ to the dimensionless form $v(s,\tau)$ with $s = x/\sigma\sqrt{2}$, $\tau = 2\omega t$, and we have compared it to the dimensionless potential $v(s)$ associated to the stationary distribution.

This behavior, that has already been observed in Ref. 18 in a similar context, has its origin in the fact that initially (until the time $t = 0$) we have a stationary state characterized by a probability density $\rho(x)$ and a velocity field $v_{(+)}(x)$, and then suddenly, in order to activate the Ornstein–Uhlenbeck decay, we impose to the same $\rho(x)$ to be embedded in the different velocity field $v^{ou}_{(+)}(x)$ which drags it toward the new, stationary and halo-free Gaussian distribution. This discontinuous change of the forward velocity is responsible for the remarked discontinuous change in the potential. We have therefore produced a transition example which starts with a sudden, discontinuous kick. At present this could have just a mathematical meaning since it would be difficult to implement it physically. However in many instances discontinuous models can be relevant as simplification of more complicated processes (as for example in rigid, instantaneous classical collisions disregarding interaction details): here in particular an impulsive external field turned up very quickly could well approximate our instantaneous change. Moreover we claim that at least in principle it would also be possible, although somewhat difficult, to construct transitions that evolve smoothly also for $t \to 0^+$ by taking into account a continuous and smooth modification of the initial velocity field into the final one.

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