Realistic physical origin of the quantum observable operator algebra in the frame of the causal stochastic interpretation of quantum mechanics: The relativistic spin-zero case

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The deduction by Guerra and Marra of the usual quantum operator algebra from a canonical variable Hamiltonian treatment of Nelson's hydrodynamical stochastic description of real nonrelativistic Schrödinger waves is extended to the causal stochastic interpretation given by Guerra and Ruggiero and by Vigier of relativistic Klein-Gordon waves. A specific representation shows that the Poisson brackets for canonical hydrodynamical observables become "averages" of quantum observables in the given state. Stochastic quantization thus justifies the standard procedure of replacing the classical particle (or field) observables with operators according to the scheme $p_\mu \to -i\hbar\partial_\mu$ and $L_{\mu\nu} \to -i\hbar(x_\mu\partial_\nu - x_\nu\partial_\mu)$.

I. INTRODUCTION

In a recent paper Guerra and Marra[1] have shown that the quantum observable operator algebra can be deduced from a stochastic variational principle with a suitable form of the stochastic action Lagrangian and Hamiltonian. This remarkable result is based on Nelson's nonrelativistic hydrodynamical stochastic analysis of Schrödinger's equation[2] and the fact that the corresponding associated Madelung-Hamilton-Jacobi and continuity equations can be interpreted as canonical Hamilton equations for a suitable symplectic system with phase space described by the density and phase action fields. As one also knows this Madelung fluid (or the corresponding de Broglie-Schrödinger wave) represents in the stochastic interpretation of quantum mechanics a real physical field (Einstein's *Gespensterwellen* or de Broglie's pilot waves) surrounding effective timelike motions of quantum particles in M_4.

The aim of the present work is to extend this nonrelativistic theory to the case of quantum operators operating on the spin-zero Klein-Gordon waves $\psi(x^\mu)$. This is evidently possible since Guerra and Ruggiero,[3] Vigier,[4] and Kyprianidis *et al.*[5] generalizing Nelson's approach have given a complete relativistic description of the corresponding subquantal relativistic random behavior. This model justifies the Klein-Gordon equation in an entirely realistic stochastic interpretation of quantum mechanics which rests on the simultaneous existence of real space-time particle trajectories (combining average timelike drift motions with random jumps at the velocity of light) and surrounding real quantum fields represented by the ψ wave.

To carry out the relativistic extension, we shall in Sec. II briefly recall the basic assumptions of Guerra on control theory and give the corresponding relativistic stochastic generalization of his formalism.

In Sec. III we shall describe the symplectic structure in phase space, give the explicit forms of the new Lagrangian and Hamiltonian, and develop the canonical theory in various representations. In Sec. IV we show how the usual quantum operator algebra results from the invariant Poisson subalgebra and give some examples of this result for given observables, including a justification of the well-known result that the quantization procedure implies the replacement of classical particle (or field) observables by operators according to the scheme $p_\mu \to -i\hbar\partial_\mu$. In Sec. V we give a physical justification for the quantization of action, following a proposal of Bohm.

To summarize, the aim of our paper is first to extend to relativity the treatment of Guerra, and second to underpin this particular formalism with a physical model in which particles are viewed as solitons surrounded by real waves propagating on a stochastic background.

II. RELATIVISTIC STOCHASTIC VARIATIONAL PRINCIPLE

To obtain the relativistic extension of the stochastic variational principle in control theory of Guerra *et al.*,[6] we have first of all to take into account the difficulties involved in defining the Markov property for relativistic processes. In fact two considerations are important for this:

(a) It has been shown[7] that if we impose on the position $x^\mu(\tau)$ in Minkowski spacetime $M^4, \tau \in \mathbb{R}$ (proper time) the condition that the future ($\tau' \geq \tau$) and the past ($\tau' \leq \tau$) are independent if the present ($\tau' = \tau$) is known, then we obtain a trivial expression for $x^\mu(\tau)$ namely, $cu^\mu(\tau - \tau_0)$ with u^μ, τ_0 constant.

(b) Guerra and Ruggiero[3] have adopted an alternative formulation of the Markov property: If σ is a spacelike three-dimensional surface in M^4 and $A_\sigma^\pm$ are the regions of M^4, respectively, in the future and the past with respect to σ, then given a diffusion process $x^\mu(\tau)$ in any region A of M^4 any event can be controlled just by looking at the parts of trajectories in this region A. Introducing the conditional expectations $E_\sigma, E_\sigma^\pm$ we can express the Markov property by writing

$$E_\sigma^+ E_\sigma^- = E_\sigma^- E_\sigma^+ = E_\sigma \ .$$

Furthermore, displacements Δx along the trajectories can be invariantly characterized by their spacelike or timelike nature $[(\Delta x)^2 \leq 0$ or $\geq 0]$, and $x^\mu(\tau)$ can be conceived of as a random process with invariant density $\rho(x,\tau)$, $x \in M^4$, $\tau \in \mathbb{R}$. By denoting $\Delta^\pm x^\mu = \pm[x^\mu(\tau \pm \Delta\tau) - x^\mu(\tau)]$, $\Delta\tau > 0$ and using the conditional expectation values we can define the forward/backward velocity vector fields $b_\pm{}^\mu$ as limits $\Delta\tau \to 0^+$:

$$E\left[\frac{\Delta^\pm x^\mu}{\Delta\tau}\middle| x(\tau)=x, (\Delta^\pm x)^2 \geq 0\right] + E\left[\frac{\Delta^\mp x^\mu}{\Delta\tau}\middle| x(\tau)=x, (\Delta^\mp x)^2 \leq 0\right] \to b_\pm^\mu(x,\tau) \ . \tag{2.1}$$

In evaluating the above expression we notice that it contains both timelike and spacelike contributions, thus indicating the fundamentally nonlocal character of relativistic quantum motions. Figure 1 illustrates the configuration of Eq. (2.1) where the forward velocity b_+^μ at point A is composed of a timelike and a spacelike part, b_{+t}^μ and b_{+s}^μ, respectively. The apparent spacelike motion is shown in Ref. 5 to be related to particle/antiparticle transitions that establish the nonlocal correlations between spacelike-separated elements. From this we deduce that the specific definition of the relativistic Markov property by Guerra and Ruggiero[3] is exactly that which is appropriate to the reproduction of the essentially nonlocal character of relativistic quantum mechanics, a fact that rests on the particle/antiparticle transition processes always present in the relativistic theory (see also Refs. 4 and 5).

With the already established relations we can define a drift velocity $v^\mu = \frac{3}{2}(b_+^\mu + b_-^\mu)$ that yields a conservation equation for the scalar density

$$\frac{\partial\rho}{\partial\tau} + \partial_\mu(\rho v^\mu) = 0 \tag{2.2}$$

and an osmotic velocity $u^\mu = \frac{1}{2}(b_+^\mu - b_-^\mu)$ which is expressed in terms of the density ρ as follows:

$$u_\mu = -(\hbar/2m)\partial_\mu \ln\rho \ , \tag{2.3}$$

where the diffusion coefficient $(\hbar/2m)$ is derived not by analogy with quantum mechanics but from de Broglie's particle-oscillator model (see Ref. 8).

Following Guerra and Ruggiero[3] we can define a forward/backward derivative for any function of a stochastic process $x^\mu(\tau)$,

$$D_\pm = \frac{\partial}{\partial\tau} + b_\pm^\mu \partial_\mu \mp (\hbar/2m)\Box \ , \tag{2.4}$$

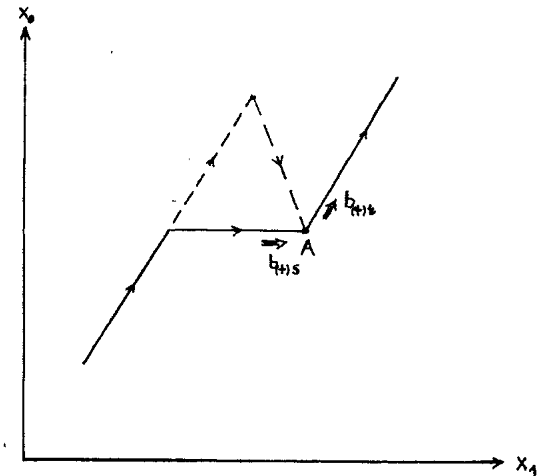

FIG. 1. The construction of apparent spacelike motions out of an underlying particle-antiparticle transition structure (dashed lines).

and from this definition we can write for the drift and osmotic derivatives D, δD

$$D = \tfrac{1}{2}(D_+ + D_-) = \frac{\partial}{\partial\tau} + v_\mu \partial^\mu \ ,$$
$$\delta D = \tfrac{1}{2}(D_+ - D_-) = u_\mu \partial^\mu - (\hbar/2m)\Box \ . \tag{2.5}$$

Having established this scheme, we now proceed to the relativistic generalization of Guerra and Morato's[6] stochastic action functionals, seeking a stochastic simulation of relativistic quantum behavior. A straightforward extension of their treatment can indeed be achieved by introducing the following relativistic Lagrangian field:

$$\mathcal{L}(x,\tau) = \tfrac{1}{2} m b_+^\mu(x,\tau) b_-^\mu(x,\tau) \ , \tag{2.6}$$

where $b_+^\mu(x,\tau)$ is considered as the control field and $b_-^\mu(x,\tau)$ can be defined through Eq. (2.3), i.e.,

$$b_{-\mu} = b_{+\mu} + (\hbar/m)\partial_\mu \ln\rho \ . \tag{2.7}$$

The average stochastic action can now be defined as

$$A = \int_{\tau_0}^{\tau_1} E(\mathcal{L}(x(\tau),\tau))d\tau = \int_{\tau_0}^{\tau_1}\int \mathcal{L}(x,\tau)\rho(x,\tau)dx d\tau \ , \tag{2.8}$$

which is the action expended by the control field in moving the system from proper time τ_0 to τ_1 for some initial distribution ρ_0.

Using standard techniques of integration by parts and Eq. (2.7) we can prove that

$$A = \tfrac{1}{2} m \int_{\tau_0}^{\tau_1}\int [b_{+\mu} b_+^\mu - (\hbar/m)\partial_\mu b_+^\mu]\rho(x,\tau)dx\, d\tau$$

which enables us to introduce the forward Lagrangian

$$\mathcal{L}_+(x,\tau) = \tfrac{1}{2} m [b_{+\mu} b_+^\mu - (\hbar/m)\partial_\mu b_+^\mu] \tag{2.9}$$

with the property $E(\mathcal{L}(x,\tau)) = E(\mathcal{L}_+(x,\tau))$. In addition, we can introduce an action functional

$$I(x,\tau;\tau_2,b_+)=-\int_{\tau}^{\tau_2}\int \mathscr{L}_+(x',\tau')p(x',\tau';x,\tau)dx'd\tau' , \quad (2.10)$$

where $p(x',\tau';x,\tau)$, $\tau'\geq\tau$ is the transition probability density and show that

$$(D_+I)(x,\tau)=\mathscr{L}_+(x,\tau) \quad (2.11)$$

with the boundary condition $I(\cdot,\tau_2)=0$. By analogy with Ref. 6 we introduce a new action functional with additional end-point contributions

$$J(x,\tau;\tau_1,S_1;v_+)=I(x,\tau;\tau_1;v_+)+E(S_1(x(\tau_1))) \quad (2.12)$$

obeying the transport equation $(D_+J)(x,\tau)=\mathscr{L}_+(x,\tau)$ and the boundary condition $J(\cdot,\tau_1)=S_1(\cdot)$. Finally, following the demonstration presented in Ref. 6 we introduce a variational principle for the control field v_+ and prove that

$$v_\mu(x,\tau)=(1/m)\partial_\mu S(x,\tau) , \quad (2.13)$$

where $S\equiv J(x,\tau;\tau_1,S_1;v_+)$ is the value of J for the stationary physical control field v_+. Furthermore since S coincides with J for the stationary field we have from Eq. (2.11)

$$(D_+S)(x,\tau)=\tfrac{1}{2}mb_{+\mu}b_+^\mu-(\hbar/2)\partial_\mu b_+^\mu . \quad (2.14)$$

Introducing the explicit form of D_+ from Eq. (2.4) and b_+^μ as $b_+^\mu=v^\mu+u^\mu$ and assuming in agreement with a tentative suggestion of Feynman[9] that $S(x^\mu,\tau)=S(x^\mu)-\tfrac{1}{2}mc^2\tau$, we obtain

$$m^2c^2-\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P+\hbar^2\Box P=0 , \quad P=\tfrac{1}{2}\ln\rho \quad (2.15)$$

which constitutes a Hamilton-Jacobi equation with stochastic corrections. This equation (2.15) together with Eqs. (2.2)—(2.13) and the ansatz

$$\psi=\sqrt{\rho(x)}\exp[(i/\hbar)S(x)] =\exp[P(x)+(i/\hbar)S(x)] \quad (2.16)$$

combine to yield the Klein-Gordon equation

$$\left[\Box+\frac{m^2c^2}{\hbar^2}\right]\psi=0 . \quad (2.17)$$

We have thus shown that the stochastic action functional, varied with respect to the stochastic control field v_+, yields the relativistic scalar particle Klein-Gordon equation of relativistic quantum mechanics. In the next section we will establish the relation of this variational principle with the standard volume variation of Lagrangian and Hamiltonian fields in the relativistic regime.

III. THE SYMPLECTIC STRUCTURE IN PHASE SPACE

Having now established an approach to relativistic Markov processes via a stochastic variational principle, we can proceed to examine the problem not in terms of a particle structure but by taking as basic variables the hydrodynamical fields given by the density $\rho(x,\tau)$ and the phase function $S(x,\tau)$. That is, one can prove that ρ and S constitute a system of canonical variables in relation to the two equations of motion, i.e., Eqs. (2.2) and (2.14). To show that this is true we proceed in the same way as Ref. 1. For this we construct a phase space Γ specified by fields ρ and S acting as canonical variables. The field label is $x\in M^4$ and we allow an explicit proper-time dependence. One should however remark at this point that, while in the classical case the time t is a universal external parameter, in the relativistic case τ is not an independent parameter since $d\tau^2=dx^\mu dx_\mu$, and in fact is defined along a certain trajectory. However, we shall assume that τ is fixed as the proper time only after the variational process is performed and an explicit solution of the equations of motion is constructed. Moreover, since the proper-time parameters on the different trajectory solutions of the same equations of motion for the one-particle problem can easily be seen to be independent, we can treat τ for the purposes of variations as a universal parameter. Hence, it attains the same status as the physical time t in the nonrelativistic theory.

Bearing these considerations in mind we introduce in Γ by analogy with Ref. 1 a symplectic structure given by the two-form

$$\omega_2(\delta\rho,\delta S;\delta'\rho,\delta'S)=\int [\delta\rho(x)\delta'S(x)-\delta'\rho(x)\delta S(x)]dx , \quad (3.1)$$

where δ and δ' are two generic systems of increments for the phase-space variables, which implies the relations

$$\{\rho(x),S(x')\}=\delta(x-x') \quad \{\rho,\rho\}=\{S,S\}=0 \quad (3.2)$$

if the Poisson brackets for generic functions $\mathscr{A}(\rho,S),\mathscr{B}(\rho,S)$ on phase space are defined as

$$\{\mathscr{A},\mathscr{B}\}=\int\left[\frac{\delta\mathscr{A}}{\delta\rho(x)}\frac{\delta\mathscr{B}}{\delta S(x)}-\frac{\delta\mathscr{A}}{\delta S(x)}\frac{\delta\mathscr{B}}{\delta\rho(x)}\right]dx . \quad (3.3)$$

In this context $\delta/\delta\rho$, $\delta/\delta S$ are the functional ρ,S derivatives, respectively.

We now introduce a relativistic Lagrangian of the form

$$\mathscr{L}=\frac{mc^2\hbar}{i}\left[\psi^*\frac{\partial\psi}{\partial\tau}-\psi\frac{\partial\psi^*}{\partial\tau}\right]+\hbar^2c^2\partial_\mu\psi^*\partial_\mu\psi \quad (3.4)$$

which can be rewritten with $\psi=\exp[P+(i/\hbar)S]$ as [with a scaling constant $(2mc^2)^{-1}$]

$$\mathscr{L}=\rho\frac{\partial S}{\partial\tau}+\frac{\rho}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P) . \quad (3.5)$$

One can check immediately that ρ and S are canonical variables of the Lagrangian by writing down the Euler-Lagrange equations with respect to ρ and S. In fact, the variation with respect to ρ yields

$$\frac{\delta\mathscr{L}}{\delta\rho}=\frac{\partial\mathscr{L}}{\partial\rho}-\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\rho)}\right]=0$$

or equivalently

$$\frac{\partial S}{\partial\tau}+\frac{1}{2m}\partial_\mu S\partial^\mu S-\frac{\hbar^2}{2m}\partial_\mu P\partial^\mu P-\frac{\hbar^2}{2m}\Box P=0 . \quad (3.6)$$

The variation with respect to S yields

$$\frac{\delta\mathcal{L}}{\delta S}=\frac{\partial\mathcal{L}}{\delta S}-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu S)}\right]-\partial_\tau\left[\frac{\partial\mathcal{L}}{\partial(\partial_\tau S)}\right]=0$$

or equivalently

$$\frac{\partial\rho}{\partial\tau}+\partial_\mu\left[\rho\frac{\partial_\mu S}{m}\right]=0\ . \tag{3.7}$$

One can easily check that in order to convert Eq. (3.6) to the Hamilton-Jacobi-type equation and Eq. (3.7) to the continuity equation of the ordinary Klein-Gordon theory one simply has to postulate in agreement with the suggestion of Feynman that the canonical variables ρ and S have the form

$$\rho(x,\tau)=\rho(x),\quad S(x,\tau)=S(x)-\tfrac{1}{2}mc^2\tau\ . \tag{3.8}$$

Equations (3.6) and (3.7) yield, respectively,

$$\Box P+\partial_\mu P\partial^\mu P-\frac{1}{\hbar^2}\partial_\mu S\partial^\mu S+\frac{m^2c^2}{\hbar^2}=0\ , \tag{3.9}$$

$$\Box S+2\partial_\mu S\partial^\mu P=0\ , \tag{3.10}$$

which constitute the real and imaginary parts of the Madelung decomposition of the ordinary Klein-Gordon equation (2.17).

A further step consists in showing that one can construct a Hamiltonian as a phase-space function for which the Hamilton equations of motion appear in canonical form. Two routes can be chosen for this depending on whether we choose ρ or S as the analog of the canonical position variable. We will briefly treat both of them starting with

(a) The $S\equiv q$ and $\rho\equiv p$ version for which $\mathcal{H}$ takes the form

$$\mathcal{H}(\rho,S)=\int\left[\rho\frac{\partial S}{\partial\tau}-\mathcal{L}\right]dx$$
$$=-\int\frac{\rho}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)dx\ . \tag{3.11}$$

Then the equations of motion read in canonical form

$$\frac{\partial S}{\partial\tau}=\{S,\mathcal{H}\}=\frac{\delta\mathcal{H}}{\delta\rho}$$

and (3.12)

$$\frac{\partial\rho}{\partial\tau}=\{\rho,\mathcal{H}\}=-\frac{\delta\mathcal{H}}{\delta S}$$

and reproduce Eqs. (3.6) and (3.7) or with (3.8) Eqs. (3.9) and (3.10), respectively.

(b) Correspondingly, the second version ($S\equiv p$ and $\rho\equiv q$) yields a Hamiltonian

$$\mathcal{H}(\rho,S)=\int\left[S\frac{\partial\rho}{\partial\tau}-\rho\frac{\partial S}{\partial\tau}-\frac{\rho}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)\right]dx \tag{3.13}$$

while the Hamilton equations of motion read

$$\frac{\partial S}{\partial\tau}=\{S,\mathcal{H}\}=-\frac{\delta\mathcal{H}}{\delta\rho}$$

and (3.14)

$$\frac{\partial\rho}{\partial\tau}=\{\rho,\mathcal{H}\}=\frac{\delta\mathcal{H}}{\delta S}$$

which reproduces again Eqs. (3.6) and (3.7) or with (3.8) Eqs. (3.9) and (3.10). In an appendix we examine the nonrelativistic limit of the above expressions.

Now, since ρ and S have been established as canonical variables of the relativistic Hamiltonian and Lagrangian, we can perform canonical transformations in phase space exactly as in Ref. 1. In fact, we can introduce a ψ_1,ψ_2 representation with

$$\psi_1=\sqrt{\rho}\cos(S/\hbar),\quad \psi_2=\sqrt{\rho}\sin(S/\hbar)$$

and show that the symplectic structure associated with ψ_1,ψ_2 is the same as that associated with ρ,S. But far more important is the equivalent representation defined in terms of the wave function and its complex conjugate:

$$\psi=\sqrt{\rho}\exp[(i/\hbar)S],\quad \psi^*=\sqrt{\rho}\exp[-(i/\hbar)S]$$

for which all the relations established by Guerra and Marra[1] in the nonrelativistic case hold here, i.e.,

$$\int\rho\delta S\,dx=\tfrac{1}{2}i\hbar\int(\psi\delta\psi^*-\psi^*\delta\psi)dx\ ,$$
$$\{\mathcal{A},\mathcal{B}\}=\frac{1}{i\hbar}\int\left[\frac{\delta\mathcal{A}}{\delta\psi}\frac{\delta\mathcal{B}}{\delta\psi^*}-\frac{\delta\mathcal{A}}{\delta\psi^*}\frac{\delta\mathcal{B}}{\delta\psi}\right]dx\ , \tag{3.15}$$

and

$$\{\psi(x),\psi^*(x')\}=\delta(x-x')/i\hbar\ ,$$
$$\{\psi,\psi\}=\{\psi^*,\psi^*\}=0\ .$$

We shall now seek an expression for the Hamiltonian operator $H_{\rm op}$ of the Klein-Gordon theory such that the field Hamiltonian $\mathcal{H}$ introduced in Eqs. (3.11) or (3.13) can be expressed as

$$\mathcal{H}=\mathcal{H}(\psi,\psi^*)=\langle\psi,H_{\rm op}\psi\rangle=\int\psi^*(x)H_{\rm op}\psi(x)dx\ . \tag{3.16}$$

This association of a $H_{\rm op}$ with $\mathcal{H}$ will be performed for each of the two $\mathcal{H}$ versions separately:

(a) Noting that $\mathcal{H}$ can be expressed in the following way in terms of ψ,ψ^*

$$\mathcal{H}=-\frac{\hbar^2}{2m}\int\partial_\mu\psi^*\partial^\mu\psi\,dx \tag{3.17}$$

and introducing the Hamiltonian operator in the form $H_{\rm op}=(\hbar^2/2m)\Box$, we can show that the expression

$$\frac{\hbar^2}{2m}\int\psi^*\Box\psi\,dx$$

can transformed by means of a simple integration by parts to the form of $\mathcal{H}$, a fact that establishes the following relation:

$$\mathscr{H}(\psi,\psi^*)=\int \psi^* H_{\rm op}\psi\, dx, H_{\rm op}=\frac{\hbar^2}{2m}\Box \,. \qquad (3.18)$$

This formula shows that the relativistic hydrodynamical Hamiltonian can be written as the "quantum average" of a relativistic quantum operator $\mathscr{H}_{\rm op}$ in the ψ representation. As a consequence the Hamilton equations are linear and coincide with the "relativistic Schrödinger equation" or for the given ansatz (3.8) with the Klein-Gordon equation. In fact one can write

$$\frac{\partial\psi}{\partial\tau}=\{\psi,\mathscr{H}\}=-\frac{1}{i\hbar}\frac{\delta\mathscr{H}}{\delta\psi^*}=-\frac{1}{i\hbar}H_{\rm op}\psi \qquad (3.19)$$

and with $\psi=\psi(x)\exp[-(i/\hbar)\frac{1}{2}mc^2\tau]$, this reduces to Eq. (2.17).

(b) Correspondingly, the second form of $\mathscr{H}$ can be written in terms of ψ,ψ^* if we remark that we can work with a new Hamiltonian (equivalent with respect to variations)

$$\mathscr{H}=-\int \rho\left[\frac{2\partial S}{\partial\tau}+\frac{1}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)\right]dx \qquad (3.20)$$

which is expressed in terms of ψ,ψ^* as

$$\mathscr{H}=\int\left[i\hbar\left[\psi^*\frac{\partial\psi}{\partial\tau}-\psi\frac{\partial\psi^*}{\partial\tau}\right]-\frac{\hbar^2}{2m}\partial_\mu\psi^*\partial^\mu\psi\right]dx \,. \qquad (3.21)$$

Noticing the specific dependence of ψ on τ, we can establish the relation

$$\mathscr{H}(\psi,\psi^*)=\int\psi^*\left[2i\hbar\frac{\partial}{\delta\tau}+\frac{\hbar^2}{2m}\Box\right]\psi\, dx \qquad (3.22)$$

from which we immediately identify $H_{\rm op}=2i\hbar\partial/\partial\tau+(\hbar^2/2m)\Box$. Following the steps of the demonstration under point (a) we can show

$$\frac{\partial\psi}{\partial\tau}=\{\psi,\mathscr{H}\}=\frac{1}{i\hbar}\frac{\delta\mathscr{H}}{\delta\psi^*}=\frac{1}{i\hbar}H_{\rm op}\psi \,. \qquad (3.23)$$

This relation together with the known ansatz for ψ reduces again to the Klein-Gordon equation.

Finally we wish to stress the remark made in Ref. 1 that it is the specific choice of the Lagrangian (and consequently the deduced Hamiltonian) that is the origin of the linearization of the Hamilton equation which produces the relativistic Schrödinger equation and furthermore, with the assumed form of ψ, the Klein-Gordon equation. It would of course be interesting to introduce, for example, a relativistic spinor Lagrangian and establish the relation between the canonical variable formalism and the operator calculus in the area of relativistic fermion theories. We shall discuss this elsewhere.

IV. DEDUCTION OF THE QUANTUM RELATIVISTIC OPERATOR ALGEBRA

By analogy with Ref. 1 let us introduce observables in phase space which in the ψ,ψ^* representation take the form

$$\mathscr{A}(\rho,S)=\int\int\psi^*(x)A(x,x')\psi(x')dx\,dx' \,, \quad x,x'\in M^4 \qquad (4.1)$$

with $A^*(x,x')=A(x',x)$, since $\mathscr{A}$ should be real, where the corresponding operator A is defined by

$$(A\psi)(x)=\int A(x,x')\psi(x')dx' \,. \qquad (4.2)$$

Since the properties of the algebra a whose elements are the bilinear observables defined by Eq. (4.1) are independent of the dimensions of the space we can simply transpose the results elaborated in Ref. 1 for the nonrelativistic case into the present context. Among these we state the property that any $\mathscr{A}\in a$ can be written in the form Eq. (4.1) [$\mathscr{A}(\rho,S)=\langle\psi,A\psi\rangle$, i.e., the Hilbert-space scalar product] for some self-adjoint operator A and, most importantly, the relation between Poisson brackets and commutators:

$$\{\mathscr{A},\mathscr{B}\}=\frac{1}{i\hbar}\int\int\psi^*(x)[A,B](x,x')\psi(x')dx\,dx' \,, \qquad (4.3)$$

where $[\cdot,\cdot]$ is the commutator of two operators. Since $\{\mathscr{A},\mathscr{B}\}$ retains the form of Eq. (4.1) we deduce that a is closed under Poisson-bracket pairing. Therefore, we find in the relativistic domain that the Poisson brackets are the relativistic "quantum average," in the above sense, of the quantum commutator.

An observable $\mathscr{A}$ generates infinitesimal canonical transformations on any $\mathscr{B}\in a$ according to the usual expression

$$\delta\mathscr{B}=\epsilon\{\mathscr{B},\mathscr{A}\} \,, \qquad (4.4)$$

where ϵ is a set of infinitesimal parameters. The remainder of this section is devoted to the construction of some characteristic examples of observables in the algebra a, bringing out that generators of infinitesimal canonical transformations are indeed associated with the expected quantum operators. We have shown already above how the Hamiltonian generates motions in proper time.

(a) Phase changes. Consider the generator $Q^\mu=\int\rho(x)x^\mu dx$. Its infinitesimal effect on ρ,S for a small vector a^μ is given by $\delta\rho(x)=0$, $\delta S(x)=-a_\mu x^\mu$. Constant phase transformations (under which the Hamiltonian is invariant) are then generated by $\int\rho(x)dx$. It is easy to see that $Q^\mu=\langle\psi,q^\mu_{\rm op}\psi\rangle$, with $q^\mu_{\rm op}=x^\mu$.

(b) Spacetime translations. The infinitesimal change $x^\mu\to x'^\mu=x^\mu-a^\mu$ induces in $\rho(x)$ the change $\delta\rho(x)=-a^\mu\partial_\mu\rho$. The corresponding infinitesimal canonical generator is $P_\mu(\rho,S)=\int\rho(x)\partial_\mu S\,dx$ as follows by evaluating $\delta\rho(x)=a^\mu\{\rho,P_\mu\}$. In terms of the ψ,ψ^* representation we obtain $P_\mu(\rho,S)=\langle\psi,P_{{\rm op}\mu}\psi\rangle$, where $P_{{\rm op}\mu}=(\hbar/i)\partial_\mu$ as expected. Similar results may be obtained by considering the transformation of $S(x)$.

(c) Lorentz rotations Under infinitesimal Lorentz transformations

$$x'^\mu=x^\mu+\epsilon^\mu{}_\nu x^\nu, \quad \epsilon_{\mu\nu}=-\epsilon_{\nu\mu} \qquad (4.5)$$

we have

$$\delta\rho(x)=\tfrac{1}{2}\epsilon^{\mu\nu}(x_\nu\partial_\mu-x_\mu\partial_\nu)\rho \,. \qquad (4.6)$$

By evaluating $\delta\rho(x)=\epsilon^{\mu\nu}\{\rho,\mathscr{L}_{\mu\nu}\}$ it is easy to see that the infinitesimal generator of canonical Lorentz transformations is

$$\mathscr{L}_{\mu\nu}(\rho,S)=\int \rho(x)(x_\mu\partial_\nu S-x_\nu\partial_\mu S)dx \ . \qquad (4.7)$$

Moreover, the corresponding quantum angular momentum operator is given by

$$\mathscr{L}_{\mu\nu}(\psi,\psi^*)=\langle\psi,L_{\text{op}\mu\nu}\psi\rangle \ ,$$
$$L_{\text{op}\mu\nu}=(\hbar/i)(x_\mu\partial_\nu-x_\nu\partial_\mu) \ . \qquad (4.8)$$

We thus see how we can deduce the form of the quantum operators for the spacetime causality group by first discovering the infinitesimal canonical generators on the hydrodynamic variables, and then passing to the ψ,ψ^* representation.

Finally, we have the following Poisson-bracket relations, corresponding to the commutation relations of the Poincaré Lie group:

$$\{P_\mu,P_\nu\}=0, \quad \{P_\mu,\mathscr{L}_{\nu\sigma}\}=\eta_{\mu\nu}P_\sigma-\eta_{\mu\sigma}P_\nu \ ,$$
$$\{\mathscr{L}_{\mu\nu},\mathscr{L}_{\sigma\rho}\}=\eta_{\mu\rho}\mathscr{L}_{\nu\sigma}+\eta_{\nu\sigma}\mathscr{L}_{\mu\rho}-\eta_{\mu\sigma}\mathscr{L}_{\nu\rho}-\eta_{\nu\rho}\mathscr{L}_{\mu\sigma}.$$

V. QUANTIZATION OF THE STOCHASTIC ACTION

The preceding introduction of stochastic canonical variables opens new vistas into possible deeper subquantal mechanisms which could justify the quantization mechanism. Following Bohm's analysis[10] what we now call "particles" are relatively stable and conserved excitations ("pilot waves" plus solitons[11]) on, the top of a real covariant vacuum (random fields) of the Dirac type.[12] Such particles will be registered only at the large-scale level since all known apparatus are only sensitive to those features of the field that will last for some time but not to those features that fluctuate rapidly. Thus the "vacuum" will produce no visible effect at the large-scale level, since its fields will cancel themselves out on the average and space will be effectively "empty" for every large-scale process, exactly like a perfect crystal lattice or a superfluid is effectively empty for an electron in the lowest band; even though the space is full of atoms.

In such a stochastic model our new canonical variables deal with some kind of average field quantities over small regions of space and time. We thus assume that a group of such average quantities would, at least, within some approximation, determine themselves independently of the infinitely complex fluctuations inside the associated regions of space, and thus define approximate average field laws associated with certain levels of size. Owing to deeper level fluctuations and, as in the case of the Brownian motion of a particle, these fluctuations will determine not only random jumps at the velocity of light, but also a probability distribution,

$$dP=P(\phi_1,\ldots,\phi_k,\ldots)d\phi_1\cdots d\phi_k\cdots \qquad (5.1)$$

which yields the mean fraction of the time in which the variables $\phi_1,\ldots,\phi_k,\ldots$ representing the mean fields in the regions $1,2,\ldots,k,\ldots$, respectively, will be in the range $d\phi_1,\ldots,d\phi_k,\ldots$. From this Bohm has derived

(1) a physical explanation for the quantization of action,

(2) a new stochastic realistic model for Heisenberg's uncertainty relations.

Bohm's results can be summarized as follows:

(1) Since we now start from a many-body Dirac-type ether model one can describe its collective large-scale behavior by collective coordinates which are an approximately self-determining set of symmetrical functions of the particle variables representing certain overall aspects such as collective oscillations. The collective motions are determined (within their characteristic domain of random fluctuation) by approximate constants of the motion. Since one can assume that our wave elements are comparable to "rigid" spherical shells enclosing bilocal oscillations[13] with $E=h\nu_0=m_0c^2$ in their rest frame we see that these collective coordinates describe nearly harmonic oscillations where the constants of the motion are the amplitudes of the oscillations and their initial phases. Such collective coordinates can be defined through the canonical transformations

$$P_k=\frac{\partial S}{\partial q_k}(q_1,\ldots,q_k,\ldots J_1,\ldots,J_n) \ ,$$
$$Q_p=\frac{\partial S}{\partial J_p}(q_1,\ldots,q_k,\ldots J_1,\ldots,J_n) \ , \qquad (5.2)$$

where S is Hamilton's transformation function, P_q and q_k the momenta and coordinates of an element, J_n and Q_n the momenta of the collective degrees of freedom. If we assume that J_n are now constants of the motion (so that in the domain where the approximation is valid the Hamiltonian only depends on the J_n and not on the Q_n) the latter increase linearly with time, like angle variables.[14] Of course, because of fluctuations of variables left out of the theory, the Q will fluctuate at random over the range accessible to them.

(2) Once the constants of motion are specified relation (5.2) reduces to

$$P_k=\frac{\partial S}{\partial q_k}(q_1,\ldots,q_k,\ldots) \qquad (5.3)$$

so that the phase S we have used as canonical variable is an average action function representing collective oscillations (and the corresponding constants of the motion) on top of the chaos of harmonic oscillators.

In that sense the expression $\psi=\rho^{1/2}e^{iS/\hbar}$ does not represent a real wave but only a canonical physical change of variables from the real variables ρ and S which describe a real field distribution; the physical reality consisting in the wave element's motions (drift plus stochastic), the evolution of their associated average density ρ and the corresponding average quantum potential $U=\Box\rho^{1/2}/\rho^{1/2}$. Thus when we give a wave function we define a canonical action function $S=\hbar\,\text{Im}(\ln\psi)$ which determines the constants of the motion through the usual classical phase integrals

$$I_c=\sum_k \oint_c P_k dq_k \ , \qquad (5.4)$$

where the integrals are taken around some circuit C

representing a set of displacements δq_k (virtual or real) in the configuration space of the system represented by ρ and S or ψ.

As one knows from Eq. (5.3) one obtains

$$I_c=\oint_c \sum \frac{\partial S}{\partial q_k}\delta q_k=\delta S_c \ , \quad (5.5)$$

where δS_c is the change of S in going around the circuit C, and we deduce the quantum quantization from the assumption that ψ is a single-valued function of all the dynamical coordinates q_k so that we have

$$\delta S_c=2n\pi\hbar=nh \ , \quad (5.6)$$

where n is an integer and h a universal constant. The problem of the quantization thus reduces to the discovery of a physical model which explains why the change of S around a circuit is restricted to discrete multiples of h, i.e., why $\exp(iS/\hbar)$ is necessarily single valued.

(3) To do this one can (should) return to an idea on which de Broglie based his discovery of wave mechanics. The idea is to consider the ψ-wave elements as extended elements containing periodic inner processes (such, as shown by Souriau *et al.*[15] generally occurs in all extended space-time structures) which determine a kind of inner time for each region of space and effectively constitute a kind of local clock attached to every ψ element with a given phase ϕ. It is now quite natural to assume (a) that in its own rest frame each clock oscillates with a uniform angular frequency $h\nu_0=m_0c^2$ which is the same for all clocks and (b) that all clocks in the same neighborhood are on the average in phase with each other. In homogeneous space there can be no reason to favor one clock over another, nor can there be a favored direction in space so that we can write

$$\delta\phi=\omega_0\delta\tau \ , \quad (5.7)$$

where $\delta\tau$ is the change of the proper time of the clock and where $\delta\phi$ is independent of δx in this frame.

Then in Bohm's words "the equality of clock phases in the rest frame for a neighborhood can be understood more deeply as a natural consequence of the nonlinearity of the coupling of the neighboring clocks (implied by the general nonlinearity of the field equations). It is well known indeed that two oscillators of the same natural frequency tend to come into phase with each other when there is such a coupling.[16] Of course the relative phases ϕ will oscillate somewhat, but, in the long run and on the average these oscillations will cancel out."[10] This property implies that once a collective moving order is established (i.e., when the clock phases vary continuously over the ψ field) it cannot be easily destroyed: so that clock synchronization is maintained by the motion and should vary continuously around any closed circuit. If we then calculate the change of $\delta\phi(x,t)$ which would follow upon a virtual displacement $(\delta x_i,\delta t)$ in any frame we get

$$\delta\phi=\omega_0\delta\tau=\frac{\omega_0\delta\tau-(v_i\delta x_i)/c^2}{(1-v^2/c^2)^{1/2}} \ ; \quad (5.8)$$

Integrating around a closed circuit the corresponding change of phase $\delta\phi_c$ should then be $2n\pi$ with a positive integer if we preserve the single-valuedness of ϕ. We thus obtain

$$\oint_c \delta\phi=\omega_0\oint_c \frac{\delta t-(v_i\delta x_i)/c^2}{(1-v^2/c^2)^{1/2}}=2n\pi \quad (5.9)$$

and introducing the total clock translation energy-momentum

$$E=m_0c^2/(1-v^2/c^2)^{1/2}, \quad p_i=m_0v_i/(1-v^2/c^2)^{1/2}$$

we get

$$\oint_c (E\delta t-p_i\delta x_i)=2n\pi(m_0/\omega_0)c^2 \quad (5.10)$$

which yields the quantum quantization if $m_0c^2/\omega_0=\hbar$.

(4) Along the same lines we can obtain a realistic justification for Heisenberg's relations[10] and for the numerical value of the diffusion constant $D=\hbar/2m$.

If we want to discuss the average change of field $\Delta\phi_k$ over a small region of time Δt (just as we had to take the average also over a region of space) we have seen that the average value of the field momentum over this time interval is then

$$\langle \pi_k\rangle=a\left[\frac{\Delta\phi_k}{\Delta t}\right] , \quad (5.11)$$

where a is a universal constant of proportionality relating the field momentum to its time derivative.

If the field fluctuates in a random way then by the very definition of randomness the region over which it fluctuates during the time Δt is given by

$$\langle(\delta\phi_k)^2\rangle \geq b\Delta t \ , \quad (5.12)$$

where b represents the basic intensity of the random fluctuation; a universal constant also if the random field fluctuations are at all places at all times the same in character.

From (5.11) we deduce that π_k will also fluctuate at random over the range

$$\delta\pi_k=\frac{a\mid\delta\phi_k\mid}{\Delta t}=\frac{ab^{1/2}}{(\Delta t)^{1/2}} \quad (5.13)$$

so that

$$\delta\pi_k\cdot\delta\phi_k\geq ab \quad (5.14)$$

independently of Δt; a relation equivalent to Heisenberg's relations if $ab=h$.

If we now consider that no interaction occurs during one stochastic jump at the velocity of light the preservation of phase continuity implies that the phase varies by 2π only so that $\langle(\delta x)^2\rangle^{1/2}=\lambda=(\hbar/mc)$. When going to $v=c$ we have $\langle(\delta x)^2\rangle=c^2\Delta\tau^2$ which yields since $D=\langle(\delta x)^2\rangle/2\Delta\tau$ the diffusion constant $D=\hbar/2m$ postulated in Nelson's original paper.[17]

We conclude this section with two remarks. The first is that in all that precedes we have only considered the behavior of the ψ field (pilot-wave) elements. Their moving average equilibrium distribution associated with the average drift current is the result of the underlying stochastic field motion which generates the collective excitation ψ over the subquantal random vacuum. The second is that if we assume[13]

(a) that only the particle aspect of matter appears in quantum experiments

(b) that this particle aspect is represented by soliton waves U traveling along the current lines of flow and beating in phase with the surrounding ψ field it has been explicitly shown[5] that any arbitrary particle distribution over the field decays in time into the distribution j_0: this propensity resulting from the field fluctuations themselves results from the property that the solitonlike particles also beat in phase with the surrounding ψ field: an essential assumption suggested by de Broglie himself to justify the $E=h\nu=mc^2$ relation.[18] This H theorem established by Kyprianidis *et al.*[5] generalizes a nonrelativistic demonstration of Bohm and Vigier.[19] It explains why the ψ field also describes quantum statistics.

VI. CONCLUSION

Following closely the procedure laid down by Guerra and Marra[1] for the nonrelativistic theory, we have in this paper set up a relativistic stochastic variational principle and established that the Lorentz scalar fields ρ, S may be treated as canonical variables in phase space. Having studied the Lagrangian and Hamiltonian structure appropriate to the Klein-Gordon equation, we have gone on to show the relation with a Hamiltonian operator formalism for relativistic quantum mechanics and demonstrated that there exists a well-defined equivalence between the Poisson brackets of the hydrodynamical theory and the commutators acting in Hilbert space. We have also justified by a physical model the single-valuedness of the wave function and the value for the diffusion coefficient.

This has been relatively straightforward since, once a relativistic Markov process has been defined, the remaining construction is independent of the signature of the metric. A significant aspect of our approach is that, in contrast to much of textbook relativistic field theory, we have developed a manifestly covariant formalism which employs a scalar Hamiltonian having the dimensions of mass squared (as in the modern theory of predictive mechanics).[20] Actually, the previously established formalism of Guerra and Marra is readily applicable to any theory cast in Schrödinger Hamiltonian form and we could have developed a relativistic spin-zero theory by employing the two-component wave-function technique proposed by Feshbach and Villars,[21] and more recently discussed by us,[22] which expresses the Klein-Gordon equation in the form $i\hbar\partial\Psi/\partial t=H\Psi$ with $\langle\Psi,H\Psi\rangle=\int T_{00}d^3x$. We prefer the present approach since it is immediately relevant to the causal interpretation. Nevertheless, it does introduce an unorthodox notion of "averaging" with respect to the quantity ρ which does not define a conserved density related to probability. Such a definition of average is evidently related to the preferred form of the Lagrangian for which the Hamiltonian equations are linear.

Providing a physical basis for the use of operators and the correspondence rule deepens the causal interpretation which, while not contesting the statistical predictions of quantum mechanics, is able to raise experimental questions which would not be suggested if the notion of trajectory were absent. We note here that current neutron interferometry experiments[23] appear to provide support for the causal-trajectory interpretation, and thus cause difficulties for the orthodox interpretation. Specifically we refer to the problem of the transfer of energy due to spin-flips in one of the paths in an interferometer and the associated measurement process[24] and to the question of the validity of energy conservation on the microscopic scale.[25] These experiments will no doubt contribute to the clarification of the question of the so-called "impossible" coexistence of trajectories and wave characteristics and in the authors' opinion give evidence in favor of the stochastic interpretation of quantum mechanics.

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APPENDIX

It is of interest to examine the nonrelativistic limit of our theory. For this we have to bear in mind that τ is an arbitrary parameter which should be treated as independent of t. Furthermore in going over to the nonrelativistic limit $S\to S'-mc^2t$ since the rest energy of the particle should not appear explicitly in the formulas.

First consider the Lagrangian,

$$L^{\text{rel}}=\int\left[\rho\frac{\partial S}{\partial\tau}+\frac{\rho}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)\right]dx\ .$$

By putting $\partial S/\partial\tau=-\frac{1}{2}mc^2$, since there will be no variation with respect to τ, $\partial_\mu S\partial^\mu S\to[m^2c^2-2m\partial_t S-(\nabla S)^2]$ and $\partial_\mu P\partial^\mu P\to-(\nabla P)^2$ and we obtain

$$L^{\text{rel}}\to L=-\int\rho\left[\partial_t S+\frac{1}{2m}(\nabla S)^2+\frac{1}{2m}(\nabla P)^2\right]dx\ ,\tag{A1}$$

where dx is over the three space variables and the Lagrangian is considered for a fixed time t. A check to ensure the validity of this L by taking the Euler-Lagrange variations with respect to ρ and S as independent variables yields the Hamilton-Jacobi-type and continuity equations of the Schrödinger theory, i.e.,

$$\partial_t S+\frac{(\nabla S)^2}{2m}-\frac{\hbar^2}{2m}[(\nabla P)^2+\Delta P]=0\ ,\tag{A2}$$

$$\partial_t\rho+\nabla\cdot\left[\rho\frac{\nabla S}{m}\right]=0\ .\tag{A3}$$

Since the equivalence of this Lagrangian with the nonrelativistic stochastic one is discussed in Ref. 6 we will dispense with further remarks on this point.

The case of the Hamiltonian is a bit more complicated. As the Hamilton canonical equations show, the relativistic

Hamiltonian is a scalar and so are its variations, e.g., $\delta\mathscr{H}/\delta\rho=\partial S/\partial\tau$, while the nonrelativistic one should apparently appear as a limit of a fourth component. If we perform the limit of our canonical variation we obtain (by the same principles as introduced previously) the following:

(a) In the first version where

$$\mathscr{H}=(-\rho/2m)(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)$$

the following equation can be written as the nonrelativistic limit of $\delta\mathscr{H}/\delta\rho=\partial S/\partial\tau$:

$$\frac{\delta}{\delta\rho}\left[\int\rho\left[2\partial_t S+\frac{(\nabla S)^2}{2m}+\frac{\hbar^2}{2m}(\nabla P)^2\right]dx\right]=\partial_t S\ . \quad \text{(A4)}$$

By means of this we deduce that the Hamiltonian for the nonrelativistic limit reads

$$\mathscr{H}^{\text{rel}}\to H=\int\rho(x,t)\left[2\partial_t S+\frac{(\nabla S)^2}{2m}+\frac{\hbar^2}{2m}(\nabla P)^2\right]dx \quad \text{(A5)}$$

for a fixed time t.

(b) In the second version where

$$\mathscr{H}=\int\left[S\frac{\partial\rho}{d\tau}-\rho\frac{\partial S}{\partial\tau}-\frac{\rho}{2m}(\partial_\mu S\partial^\mu S+\hbar^2\partial_\mu P\partial^\mu P)\right]dx$$

and $\delta\mathscr{H}/\delta\rho=-\partial S/\partial\tau$ the following equation can be derived in the limit:

$$\frac{\delta}{\delta\rho}\left[\int\rho\left[\frac{(\nabla S)^2}{2m}+\frac{\hbar^2}{2m}(\nabla P)^2\right]dx\right]=-\partial_t S\ . \quad \text{(A6)}$$

This identifies the nonrelativistic limit of $\mathscr{H}$ as

$$\mathscr{H}^{\text{rel}}\to H=\int\rho\left[\frac{(\nabla S)^2}{2m}+\frac{\hbar^2}{2m}(\nabla P)^2\right]dx \quad \text{(A7)}$$

for a fixed time. This coincides with the choice of H in Ref. 1, a fact that is obvious if one notices that the choice of ρ as canonical position and S as canonical momentum has been made.

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